LAGUERRE SEMIGROUP AND DUNKL OPERATORS

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ABSTRACT. We construct a two-parameter family of actions $\omega_{k,a}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by differential-difference operators on $\mathbb{R}^N \setminus \{0\}$. Here, k is a multiplicity-function for the Dunkl operators, and $a > 0$ arises from the interpolation of the two $\mathfrak{sl}(2,\mathbb{R})$ actions on the Weil representation of $Mp(N,\mathbb{R})$ and the minimal unitary representation of $O(N+1,2)$. We prove that this action $\omega_{k,a}$ lifts to a unitary representation of the universal covering of $SL(2,\mathbb{R})$, and can even be extended to a holomorphic semigroup $\Omega_{k,a}$. In the $k \equiv 0$ case, our semigroup generalizes the Hermite semigroup studied by R. Howe ($a = 2$) and the Laguerre semigroup by the second author with G. Mano ($a = 1$). One boundary value of our semigroup $\Omega_{k,a}$ provides us with (k,a)-*generalized Fourier transforms* $\mathscr{F}_{k,a}$, which includes the Dunkl transform $\mathscr{D}_k$ ($a = 2$) and a new unitary operator $\mathscr{H}_k$ ($a = 1$), namely a Dunkl–Hankel transform. We establish the inversion formula, and a generalization of the Plancherel theorem, the Hecke identity, the Bochner identity, and a Heisenberg uncertainty relation for $\mathscr{F}_{k,a}$. We also find kernel functions for $\Omega_{k,a}$ and $\mathscr{F}_{k,a}$ for $a = 1, 2$ in terms of Bessel functions and the Dunkl intertwining operator.

CONTENTS

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1. Introduction

The classical Fourier transform is one of the most basic objects in analysis; it may be understood as belonging to a one-parameter group of unitary operators on $L^2(\mathbb{R}^N)$, and this group may even be extended holomorphically to a semigroup (the *Hermite semigroup*) $I(z)$ generated by the self-adjoint operator $\Delta - \|x\|^2$. This is a holomorphic semigroup of bounded operators depending on a complex variable z in the complex right half-plane, viz. $I(z + w) = I(z)I(w)$. The structure of this semigroup and its properties may be appreciated without any reference to representation theory, whereas the link itself is rich as was revealed beautifully by R. Howe [29] in connection with the Schrödinger model of the Weil representation.

Our primary aim of this article is to give a foundation of the deformation theory of the classical situation, by constructing a generalization $\mathscr{F}_{k,a}$ of the Fourier transform, and the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ with infinitesimal generator $\|x\|^{2-a}\Delta_k - \|x\|^a$, acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^N)$. Here Δ_k is the Dunkl Laplacian (a differential-difference operator). We analyze these operators $\mathscr{F}_{k,a}$ and $\mathscr{I}_{k,a}(z)$ in the context of integral operators as well as representation theory.

The deformation parameters in our setting consist of a real parameter a coming from the interpolation of the minimal unitary representations of two different reductive groups by keeping smaller symmetries (see Diagram 1.4), and a parameter k coming from Dunkl's theory of differential-difference operators associated to a finite Coxeter group; also the dimension N and the complex variable z may be considered as a parameter of the theory.

We point out, that already deformations with $k = 0$ are new and interpolate the minimal representations of two reductive groups $O_0(n+1, 2)^\sim$ and $Mp(n, \mathbb{R})$. Notice that these unitary representations are generated by the 'unitary inversion operator' (= $\mathscr{F}_{0,a}$ with $a = 1, 2$, up to a scalar multiplication) together with an elementary action of the maximal parabolic subgroups (see [37] and [38, Introduction]).

This article establishes the foundation of these new operators. Our theorems on (k, a)-generalized Fourier transforms $\mathscr{F}_{k,a}$ include:

– Plancherel and inversion formula (Theorems 5.1 and 5.3),
– Bochner-type theorem (Theorem 5.21),
– Heisenberg's uncertainty relation (Theorem 5.29),
– exchange of multiplication and differentiation (Theorem 5.6).

We think of the results and the methods here as opening potentially interesting studies such as:

– characterization of 'Schwartz space' and Paley–Wiener type theorem,
– Strichartz estimates for Schrödinger and wave equations,
– Brownian motions in a Weyl chamber (cf. [20]),
– analogues of Clifford analysis for the Dirac operator (cf. [48]),

working with deformations of classical operators.

In the diagram below we have summarized some of the deformation properties by indicating the limit behaviour of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$; it is seen how various previous integral transforms fit in our picture. In particular we obtain as special cases the Dunkl transform $\mathscr{D}_k$ [11] ($a = 2$, $z = \frac{\pi i}{2}$ and k arbitrary), the Hermite semigroup $I(z)$ [18, 29] ($a = 2$, $k \equiv 0$ and z arbitrary), and the Laguerre semigroup [35, 36] ($a = 1$, $k \equiv 0$ and z arbitrary). Our framework gives a new treatment even on the theory of the Dunkl transform.

The 'boundary value' of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ from $\operatorname{Re} z > 0$ to the imaginary axis gives rise to a one-parameter subgroup of unitary operators. The underlying idea may be interpreted as a descendent of Sato's hyperfunction theory [52] and also that of the Gelfand–Gindikin program [21, 27, 46, 54] for unitary representations of real reductive groups. The specialization $\mathscr{I}_{k,a}(\frac{\pi i}{2})$ will be our (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ (up to a phase factor), which reduces to the Fourier transform ($a = 2$ and $k \equiv 0$), the Dunkl transform $\mathscr{D}_k$ ($a = 2$ and k arbitrary), and the Hankel transform ($a = 1$ and $k \equiv 0$).

Yet another specialization is to take $N = 1$. This very special case contains (after some change of variables) the results on the L^2-model of the highest weight representations of the universal covering group of $SL(2, \mathbb{R})$, which was obtained by B. Kostant [40] and R. Rao [47] by letting $\mathfrak{sl}_2$ act as differential operators on the half-line (see Remark 3.32).

The secondary aim of this article is to contribute to the theory of special functions, in particular orthogonal polynomials; indeed we derive several new identities, for example, the (k, a)-deformation of the classical Hecke identity (Corollary 5.20) where the Gaussian function and harmonic polynomials in the classical setting are replaced respectively with $\exp(-\frac{1}{a}\|x\|^a)$ and polynomials annihilated by the Dunkl Laplacian. Another example is the identity (4.41), which expresses an infinite sum of products of Bessel functions and Gegenbauer functions as a single Bessel function.

In the rest of the Introduction we describe a little more the contents of this article.

In Sections 1.1 and 1.2, without any reference to representation theory, we discuss our holomorphic semigroup $\mathscr{I}_{k,a}(z)$ and (k, a)-generalized Fourier transforms $\mathscr{F}_{k,a}$ as a two-parameter deformation of the classical objects, i.e. the Hermite semigroup and the Euclidean Fourier transform.

In Section 1.3, we introduce the basic machinery of the present article, namely, to construct triples of differential-difference operators generating the Lie algebra of $SL(2,\mathbb{R})$, and see how they are integrated to unitary representations of the universal covering group.

One further aspect of our constructions is the link to minimal unitary representations. For the specific two parameters $(a,k) = (1,0)$ and $(2,0)$, we are really working with representations of much larger semisimple groups, and our deformation is interpolating the representation spaces for the minimal representations of two different groups. We highlight these hidden symmetries in Section 1.4.

Let us also note that there is in our theory a natural appearance of some symmetries of the double degeneration of the double affine Hecke algebra (sometimes called the *rational Cherednik algebra*), see Section 5.6. Here $a = 2$ and k arbitrary, and in particular, we recover the Dunkl transform.

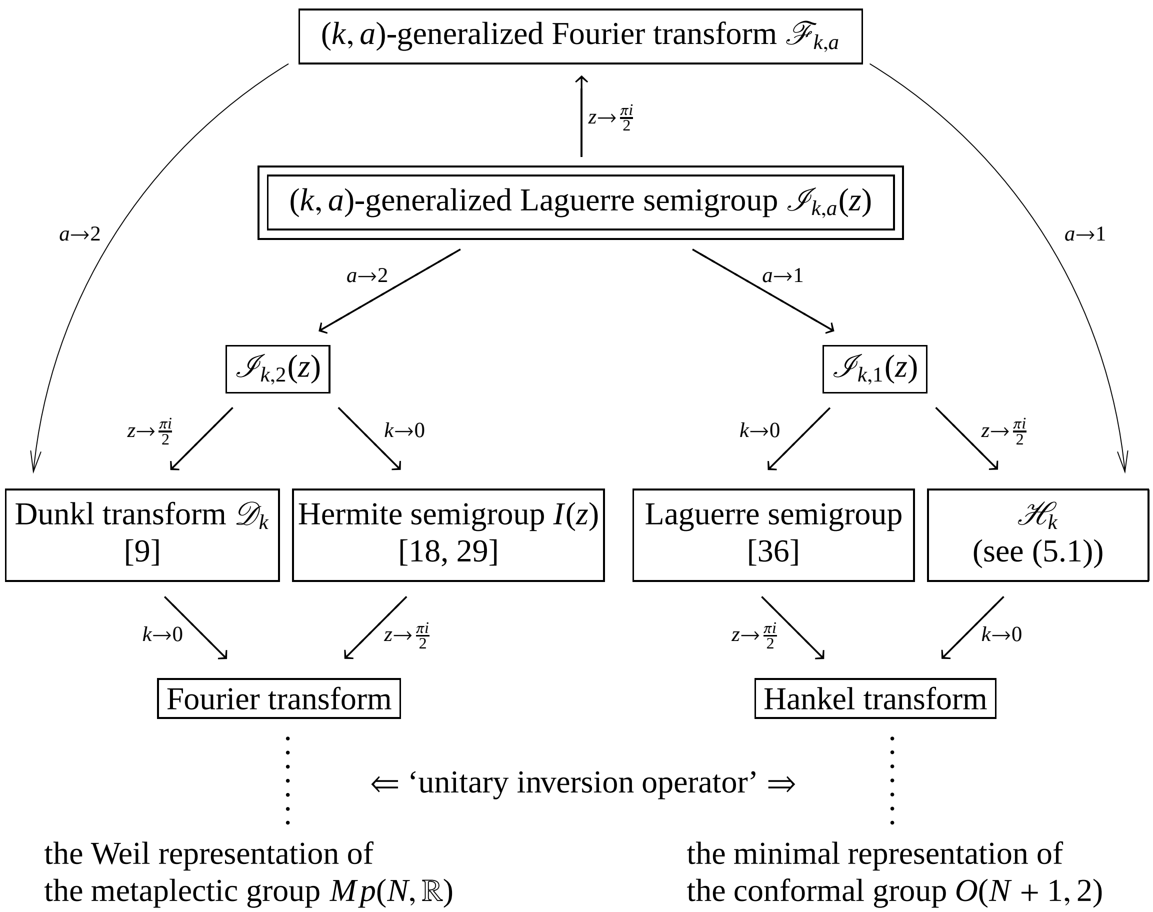

Diagram 1. Special values of holomorphic semigroup $\mathscr{I}_{k,a}(z)$

1.1. **Holomorphic semigroup $\mathscr{I}_{k,a}(z)$ with two parameters k and a.**

Dunkl operators are differential-difference operators associated to a finite reflection group on the Euclidean space. They were introduced by C. Dunkl [9]. This subject was motivated partly from harmonic analysis on the tangent space of the Riemannian symmetric spaces, and resulted in a new theory of non-commutative harmonic analysis 'without Lie groups'. The Dunkl operators are also used as a tool for investigating an algebraic integrability property for the Calogero–Moser quantum problem related to root systems [23]. We refer to [13] for the up-to-date survey on various applications of Dunkl operators.

Our holomorphic semigroup $\mathscr{I}_{k,a}(z)$ is built on Dunkl operators. To fix notation, let $\mathfrak{C}$ be the Coxeter group associated with a root system $\mathscr{R}$ in $\mathbb{R}^N$. For a $\mathfrak{C}$-invariant real function $k \equiv (k_\alpha)$ (*multiplicity function*) on $\mathscr{R}$, we write Δ_k for the Dunkl Laplacian on $\mathbb{R}^N$ (see (2.10)).

We take $a > 0$ to be a deformation parameter, and introduce the following differential-difference operator

$$\Delta_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a. \tag{1.1}$$

Here, $\|x\|$ is the norm of the coordinate $x \in \mathbb{R}^N$, and $\|x\|^a$ in the right-hand side of the formula stands for the multiplication operator by $\|x\|^a$. Then, $\Delta_{k,a}$ is a symmetric operator on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ consisting of square integrable functions on $\mathbb{R}^N$ against the measure $\vartheta_{k,a}(x)dx$, where the density function $\vartheta_{k,a}(x)$ on $\mathbb{R}^N$ is given by

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha\in\mathscr{R}} |\langle\alpha, x\rangle|^{k_\alpha}. \tag{1.2}$$

Then $\vartheta_{k,a}(x)$ has a degree of homogeneity $a - 2 + 2\langle k\rangle$, where $\langle k\rangle := \frac{1}{2}\sum_{\alpha\in\mathscr{R}} k_\alpha$ is the index of $k = (k_\alpha)$ (see (2.3)).

The (k, a)-*generalized Laguerre semigroup* $\mathscr{I}_{k,a}(z)$ is defined to be the semigroup with infinitesimal generator $\frac{1}{a}\Delta_{k,a}$, that is,

$$\mathscr{I}_{k,a}(z) := \exp\Big(\frac{z}{a}\Delta_{k,a}\Big), \tag{1.3}$$

for $z \in \mathbb{C}$ such that $\operatorname{Re} z \geq 0$. (Later, we shall use the notation $\mathscr{I}_{k,a}(z) = \Omega_{k,a}(\gamma_z)$, in connection with the Gelfand–Gindikin program.)

In the case $a = 2$ and $k \equiv 0$, the density $\vartheta_{k,a}(x)$ reduces to $\vartheta_{0,2}(x) \equiv 1$ and we recover the classical setting where

$$\Delta_{0,2} = \sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} - \sum_{j=1}^{N} x_j^2, \text{ the Hermite operator on } L^2(\mathbb{R}^N),$$

$$\mathscr{I}_{0,2}(z) = \text{the Hermite semigroup } I(z) \text{ ([18, 29])}.$$

In this article, we shall deal with a positive a and a non-negative multiplicity function k for simplicity, though some of our results still hold for "slightly-negative" multiplicity functions (see Remark 2.3). We begin with:

Theorem A (see Corollary 3.22)**.** *Suppose $a > 0$ and a non-negative multiplicity function k satisfy $a + 2\langle k\rangle + N - 2 > 0$. Then,*

1) *$\Delta_{k,a}$ extends to a self-adjoint operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*
2) *There is no continuous spectrum of $\Delta_{k,a}$.*
3) *All the discrete spectra are negative.*

We also find all the discrete spectra explicitly in Corollary 3.22.

Turning to the (k, a)-generalized Laguerre semigroup $\mathscr{I}_{k,a}(z)$ (see (1.3)), we shall prove:

Theorem B (see Theorem 3.39)**.** *Retain the assumptions of Theorem A.*

1) $\mathscr{I}_{k,a}(z)$ *is a holomorphic semigroup in the complex right-half plane* $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ *in the sense that* $\mathscr{I}_{k,a}(z)$ *is a Hilbert–Schmidt operator on* $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ *satisfying*

$$\mathscr{I}_{k,a}(z_1) \circ \mathscr{I}_{k,a}(z_2) = \mathscr{I}_{k,a}(z_1 + z_2), \quad (\operatorname{Re} z_1, \operatorname{Re} z_2 > 0),$$

and that the scalar product $(\mathscr{I}_{k,a}(z) f, g)$ *is a holomorphic function of* z *for* $\operatorname{Re} z > 0$*, for any* $f, g \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$*.*

2) $\mathscr{I}_{k,a}(z)$ *is a one-parameter group of unitary operators on the imaginary axis* $\operatorname{Re} z = 0$*.*

In Section 4.3, we shall introduce a real analytic function $\mathscr{I}(b, \nu; w; \cos\varphi)$ in four variables defined on $\{(b, \nu, w, \varphi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{C} \times \mathbb{R}/2\pi\mathbb{Z} : 1 + b\nu > 0\}$. The special values at $b = 1, 2$ are given by

$$\mathscr{J}(1, \nu; w; t) = e^{wt}, \tag{1.4}$$

$$\mathscr{J}(2, \nu; w; t) = \Gamma(\nu + \frac{1}{2})\widetilde{I}_{\nu-\frac{1}{2}}\Big(\frac{w(1+t)^{1/2}}{\sqrt{2}}\Big). \tag{1.5}$$

Here, $\widetilde{I}_\lambda(z) = (\frac{z}{2})^{-\lambda} I_\lambda(z)$ is the (normalized) modified Bessel function of the first kind (simply, I-Bessel function). We notice that these are positive-valued functions of t if $w \in \mathbb{R}$.

We then define the following continuous function of t on the interval $[-1, 1]$ with parameters $r, s > 0$ and $z \in \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\} \setminus i\pi\mathbb{Z}$ by

$$h_{k,a}(r, s; z; t) = \frac{\exp\big(-\frac{1}{a}(r^a + s^a)\coth(z)\big)}{\sinh(z)^{\frac{2\langle k\rangle + N + a - 2}{a}}} \mathscr{J}\left(\frac{2}{a}, \frac{2\langle k\rangle + N - 2}{2}; \frac{2(rs)^{\frac{a}{2}}}{a\sinh(z)}; t\right),$$

where $\langle k\rangle = \frac{1}{2}\sum_{\alpha\in\mathscr{R}} k_\alpha$ (see (2.3)).

For a function $h(t)$ of one variable, let $(\widetilde{V}_k h)(x, y)$ be a k-deformation of the function $h(\langle x, y\rangle)$ on $\mathbb{R}^N \times \mathbb{R}^N$. (This k-deformation is defined by using the Dunkl intertwining operator V_k, see (2.6)).

In the polar coordinates $x = r\omega$ and $y = s\eta$, we set

$$\Lambda_{k,a}(x, y; z) = \widetilde{V}_k(h_{k,a}(r, s; z; \cdot))(\omega, \eta).$$

For $a > 0$ and a non-negative multiplicity function k, we introduce the following normalization constant

$$c_{k,a} := \Big(\int_{\mathbb{R}^N} \exp\left(-\frac{1}{a}\|x\|^a\right)\vartheta_{k,a}(x)dx\Big)^{-1}. \tag{1.6}$$

The constant $c_{k,a}$ can be expressed in terms of the gamma function owing to the work by Selberg, Macdonald, Heckman, Opdam [45], and others (see Etingof [16] for a uniform proof).

Here is an integration formula of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$.

Theorem C (see Theorem 4.23)**.** *Suppose* $a > 0$ *and* k *is a non-negative multiplicity function. Suppose* $\operatorname{Re} z \geq 0$ *and* $z \notin i\pi\mathbb{Z}$*. Then,* $\mathscr{I}_{k,a}(z) = \exp(\frac{z}{a}\Delta_{k,a})$ *is given by*

$$\mathscr{I}_{k,a}(z) f(x) = c_{k,a} \int_{\mathbb{R}^N} f(y)\Lambda_{k,a}(x, y; z)\vartheta_{k,a}(y)dy. \tag{1.7}$$

The formula (1.7) generalizes the $k \equiv 0$ case; see Kobayashi–Mano [36] for $(k, a) = (0, 1)$, and the Mehler kernel formula in Folland [18] or Howe [29] for $(k, a) = (0, 2)$.

1.2. (k,a)**-generalized Fourier transforms** $\mathscr{F}_{k,a}$**.**
As we mentioned in Theorem B 2), the 'boundary value' of the (k,a)-generalized Laguerre semigroup $\mathscr{I}_{k,a}(z)$ on the imaginary axis gives a one-parameter family of unitary operators. The case $z = 0$ gives the identity operator, namely, $\mathscr{I}_{k,a}(0) = \mathrm{id}$. The particularly interesting case is when $z = \frac{\pi i}{2}$, and we set

$$\mathscr{F}_{k,a} := c\,\mathscr{I}_{k,a}\Big(\frac{\pi i}{2}\Big) = c\exp\Big(\frac{\pi i}{2a}(\|x\|^{2-a}\Delta_k - \|x\|^a)\Big)$$

by multiplying the phase factor $c = e^{i\frac{\pi}{2}(\frac{2\langle k\rangle+N+a-2}{a})}$ (see (5.2)). Then, the unitary operator $\mathscr{F}_{k,a}$ for general a and k satisfies the following significant properties:

Theorem D (see Proposition 3.35 and Theorem 5.6)**.** *Suppose $a > 0$ and k is a non-negative multiplicity function such that $a + 2\langle k\rangle + N - 2 > 0$.*
1) *$\mathscr{F}_{k,a}$ is a unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*
2) *$\mathscr{F}_{k,a} \circ E = -(E + N + 2\langle k\rangle + a - 2) \circ \mathscr{F}_{k,a}$.*
   Here, $E = \sum_{j=1}^N x_j\partial_j$.
3) *$\mathscr{F}_{k,a} \circ \|x\|^a = -\|x\|^{2-a}\Delta_k \circ \mathscr{F}_{k,a}$,*
   $\mathscr{F}_{k,a} \circ (\|x\|^{2-a}\Delta_k) = -\|x\|^a \circ \mathscr{F}_{k,a}$.
4) *$\mathscr{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$. Its order is $2p$ if a is of the form $a = \frac{p}{q}$, where p and q are positive integers that are relatively prime.*

We call $\mathscr{F}_{k,a}$ a (k,a)-*generalized Fourier transform* on $\mathbb{R}^N$. We note that $\mathscr{F}_{k,a}$ reduces to the Euclidean Fourier transform $\mathscr{F}$ if $k \equiv 0$ and $a = 2$; to the Hankel transform if $k \equiv 0$ and $a = 1$; to the Dunkl transform $\mathscr{D}_k$ introduced by C. Dunkl himself in [11] if $k > 0$ and $a = 2$.

For $a = 2$, our expressions of $\mathscr{F}_{k,a}$ amount to:

$$\mathscr{F} = e^{\frac{\pi i N}{4}}\exp\frac{\pi i}{4}(\Delta - \|x\|^2) \qquad \text{(Fourier transform)},$$

$$\mathscr{D}_k = e^{\frac{\pi i(2\langle k\rangle+N)}{4}}\exp\frac{\pi i}{4}(\Delta_k - \|x\|^2) \qquad \text{(Dunkl transform)}.$$

For $a = 1$ and $k \equiv 0$, the unitary operator

$$\mathscr{F}_{0,1} = e^{\frac{\pi i(N-1)}{2}}\exp\Big(\frac{\pi i}{2}\|x\|(\Delta - 1)\Big)$$

arises as the *unitary inversion operator* of the Schrödinger model of the minimal representation of the conformal group $O(N+1,2)$ (see [35, 36]). Its Dunkl analogue, namely, the unitary operator $\mathscr{F}_{k,a}$ for $a = 1$ and $k > 0$ seems also interesting, however, it has never appeared in the literature, to the best of our knowledge. The integral representation of this unitary operator,

$$\mathscr{H}_k := \mathscr{F}_{k,1} = e^{i\frac{\pi}{2}(2\langle k\rangle+N-1)}\mathscr{I}_{k,1}\Big(\frac{\pi i}{2}\Big) = e^{i\frac{\pi}{2}(2\langle k\rangle+N-1)}\exp\Big(\frac{\pi i}{2}\|x\|(\Delta_k - 1)\Big),$$

is given in terms of the Dunkl intertwining operator and the Bessel function due to the closed formula of $\mathscr{I}(b,\nu;w;t)$ at $b = 2$ (see (1.5)).

On the other hand, our methods can be applied to general k and a in finding some basic properties of the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ such as the inversion formula, the Plancherel theorem, the Hecke identity (Corollary 5.20), the Bochner identity (Theorem 5.21), and the following Heisenberg inequality (Theorem 5.29):

Theorem E (Heisenberg type inequality)**.** *Let $\| \ \|_k$ denote by the norm on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. Then,*

$$\left\| \|x\|^{\frac{a}{2}} f(x) \right\|_k \left\| \|\xi\|^{\frac{a}{2}} \mathscr{F}_{k,a} f(\xi) \right\|_k \geq \frac{2\langle k \rangle + N + a - 2}{2} \|f(x)\|_k^2$$

for any $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. The equality holds if and only if f is a scalar multiple of $\exp(-c\|x\|^a)$ *for some $c > 0$.*

This inequality was previously proved by Rösler [50] and Shimeno [53] for the $a = 2$ case (i.e. the Dunkl transform $\mathscr{D}_k$). In physics terms we may think of the function where the equality holds in Theorem E as a ground state; indeed when $a = c = 1$, $N = 3$, and $k \equiv 0$ it is exactly the wave function for the Hydrogen atom with the lowest energy.

1.3. $\mathfrak{sl}_2$-triple of differential-difference operators.

Over the last several decades, various works have been published that develop applications of the representation theory of the special linear group $SL(2, \mathbb{R})$. We mention particularly the books of Lang [41] and Howe–Tan [30], and the research papers of Vergne [57] and Howe [28]. These and other contributions show how the symmetries of $\mathfrak{sl}_2$ can offer new perspectives on familiar topics from inside and outside representation theory (character formulas, ergodic theory, Fourier analysis, the Laplace equation, etc.).

The basic tool for the present article is also the SL_2 theory. We construct an $\mathfrak{sl}_2$-triple of differential-difference operators with two parameters k and a, and then apply representation theory of $\widetilde{SL(2, \mathbb{R})}$, the universal covering group of $SL(2, \mathbb{R})$. The resulting representation is a discretely decomposable unitary representation in the sense of [33], which depends continuously on parameters a and k.

To be more precise, we introduce the following differential-difference operators on $\mathbb{R}^N \setminus \{0\}$ by

$$\mathbb{E}^+_{k,a} := \frac{i}{a}\|x\|^a, \qquad \mathbb{E}^-_{k,a} := \frac{i}{a}\|x\|^{2-a}\Delta_k, \qquad \mathbb{H}_{k,a} := \frac{2}{a}\sum_{i=1}^N x_i\partial_i + \frac{N + 2\langle k \rangle + a - 2}{a}.$$

With these operators, we have

$$a\Delta_{k,a} = i\,(\mathbb{E}^+_{k,a} - \mathbb{E}^-_{k,a}).$$

The main point here is that our operator $\Delta_{k,a}$ can be interpreted in the framework of the (infinite dimensional) representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$:

Lemma F (see Theorem 3.2)**.** *The differential-difference operators $\{\mathbb{H}_{k,a}, \mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}\}$ form an $\mathfrak{sl}_2$-triple for any multiplicity-function k and any non-zero complex number a.*

In other words, taking a basis of $\mathfrak{sl}(2, \mathbb{R})$ as

$$\mathbf{e}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we get a Lie algebra representation $\omega_{k,a}$ of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ with continuous parameters k and a on functions on $\mathbb{R}^N$ by mapping

$$\mathbf{h} \mapsto \mathbb{H}_{k,a}, \quad \mathbf{e}^+ \mapsto \mathbb{E}^+_{k,a}, \quad \mathbf{e}^- \mapsto \mathbb{E}^-_{k,a}.$$

The main result of Section 3 is to prove that the representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$ lifts to the universal covering group $\widetilde{SL(2,\mathbb{R})}$:

Theorem G (see Theorem 3.30)**.** *If $a > 0$ and k is a non-negative multiplicity function such that $a + 2\langle k\rangle + N - 2 > 0$, then $\omega_{k,a}$ lifts to a unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*

Theorem G fits nicely into the framework of discretely decomposable unitary representations [33, 34]. In fact, we see in Theorem 3.31 that the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ decomposes discretely as a direct sum of unitary representations of the direct product group $\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}$:

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \simeq \sum_{m=0}^{\infty}{}^{\oplus} \mathscr{H}_k^m(\mathbb{R}^N)\big|_{S^{N-1}} \otimes \pi\Big(\frac{2m + 2\langle k\rangle + N - 2}{a}\Big), \tag{1.8}$$

where $\mathscr{H}_k^m(\mathbb{R}^N)$ stands for the representation of the Coxeter group $\mathfrak{C}$ on the eigenspace of the Dunkl Laplacian (the space of spherical k-harmonics of degree m) and $\pi(\nu)$ is an irreducible unitary lowest weight representation of $\widetilde{SL(2,\mathbb{R})}$ of weight $\nu + 1$ (see Fact 3.27). The unitary isomorphism (1.8) is constructed explicitly by using Laguerre polynomials.

For general $N \geq 2$, the right-hand side of (1.8) is an infinite sum. For $N = 1$, (1.8) is reduced to the sum of two terms ($m = 0, 1$).

The unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ extends furthermore to a holomorphic semigroup of a complex three dimensional semigroup (see Section 3.8). Basic properties of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ defined in (1.3) and the unitary operator $\mathscr{F}_{k,a}$ can be read from the 'dictionary' of $\mathfrak{sl}(2,\mathbb{R})$ as follows:

$$\begin{aligned}
i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &\longleftrightarrow \frac{1}{a}\Delta_{k,a} \\
\exp iz\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &\longleftrightarrow \mathscr{I}_{k,a}(z) = \exp(\frac{z}{a}\Delta_{k,a}) \\
w_0 = \exp\frac{\pi}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &\longleftrightarrow \mathscr{F}_{k,a} \text{ (up to the phase factor)} \\
\mathrm{Ad}(w_0)\mathbf{e}^+ = \mathbf{e}^- &\longleftrightarrow \mathscr{F}_{k,a} \circ ||x||^a = -||x||^{2-a}\Delta_k\mathscr{F}_{k,a} \\
\mathrm{Ad}(w_0)\mathbf{e}^- = \mathbf{e}^+ &\longleftrightarrow \mathscr{F}_{k,a} \circ ||x||^{2-a}\Delta_k = -||x||^a\mathscr{F}_{k,a}.
\end{aligned}$$

1.4. **Hidden symmetries for $a = 1$ and 2.**

As we have seen in Section 1.1, one of the reasons that we find an explicit formula for the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ (and for the unitary operator $\mathscr{F}_{k,a}$) (see Section 1.1) is that there are large 'hidden symmetries' on the Hilbert space when $a = 1$ or 2.

We recall that our analysis is based on the fact that the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ has a symmetry of the direct product group $\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}$ for all k and a. It turns out that this symmetry becomes larger for special values of k and a. In this subsection, we discuss these hidden symmetries.

First, in the case $k \equiv 0$, the Dunkl Laplacian Δ_k becomes the Euclidean Laplacian Δ, and consequently, not only the Coxeter group $\mathfrak{C}$ but also the whole orthogonal group $O(N)$

commutes with $\Delta_k \equiv \Delta$. Therefore, the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{0,a}(x)dx)$ is acted on by $O(N) \times \widetilde{SL(2,\mathbb{R})}$. Namely, it has a larger symmetry

$$\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})} \subset O(N) \times \widetilde{SL(2,\mathbb{R})}.$$

Next, we observe that the Lie algebra of the direct product group $O(N) \times \widetilde{SL(2,\mathbb{R})}$ may be seen as a subalgebra of two different reductive Lie algebras $\mathfrak{sp}(N,\mathbb{R})$ and $\mathfrak{o}(N+1,2)$:

$$\mathfrak{o}(N) \oplus \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{o}(N) \oplus \mathfrak{o}(1,2) \subset \mathfrak{o}(N+1,2)$$
$$\mathfrak{o}(N) \oplus \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{o}(N) \oplus \mathfrak{sp}(1,\mathbb{R}) \subset \mathfrak{sp}(N,\mathbb{R})$$

It turns out that they are the hidden symmetries of the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{0,a}(x)dx)$ for $a = 1, 2$, respectively. To be more precise, the conformal group $O(N+1,2)_0$ (or its double covering group if N is even) acts on $L^2(\mathbb{R}^N, \vartheta_{0,1}(x)dx) = L^2(\mathbb{R}^N, \|x\|^{-1}dx)$ as an irreducible unitary representation, while the metaplectic group $Mp(N,\mathbb{R})$ (the double covering group of the symplectic group $Sp(N,\mathbb{R})$) acts on $L^2(\mathbb{R}^N, \vartheta_{0,2}(x)dx) = L^2(\mathbb{R}^N, dx)$ as a unitary representation.

In summary, we are dealing with the symmetries of the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ described below:

$$\begin{tikzcd}
 & & \boxed{O(N+1,2)} \\
\boxed{\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}} \arrow[r, "k\to 0"] & \boxed{O(N) \times \widetilde{SL(2,\mathbb{R})}} \arrow[ur, "a\to 1"'] \arrow[dr, "a\to 2"] & \\
 & & \boxed{Mp(N,\mathbb{R})}
\end{tikzcd}$$

(k, a: general)

Diagram 1.4. Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

For $a = 2$, this unitary representation is nothing but the Weil representation, sometimes referred to as the Segal–Shale–Weil representation, the metaplectic representation, or the oscillator representation, and its realization on $L^2(\mathbb{R}^N)$ is called the Schrödinger model.

For $a = 1$, the unitary representation of the conformal group on $L^2(\mathbb{R}^N, \|x\|^{-1}dx)$ is irreducible and has a similar nature to the Weil representation. The similarity is illustrated by the fact that both of these unitary representations are 'minimal representations', i.e., their annihilator of the infinitesimal representations are the Joseph ideal of the universal enveloping algebras, and in particular, they attain the minimum of their Gelfand–Kirillov dimensions.

In this sense, our continuous parameter $a > 0$ interpolates two minimal representations of different reductive groups by keeping smaller symmetries (i.e. the representations of $O(N) \times \widetilde{SL(2,\mathbb{R})}$). The (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ plays a special role in the global formula of the L^2-model of minimal representations. In fact, the conformal group $O(N+1,2)$ is generated by a maximal parabolic subgroup (essentially, the affine conformal group for the Minkowski space $\mathbb{R}^{N,1}$) and the inversion element $I_{N+1,2} = \mathrm{diag}(1,\ldots,1,-1,-1)$.

Likewise, the metaplectic group $Mp(N, \mathbb{R})$ is generated by the Siegel parabolic subgroup and the conformal inversion element. Since the maximal parabolic subgroup acts on the L^2-model on the minimal representation, we can obtain the global formula of the whole group if we determine the action of the inversion element. For the Weil representation, this crucial action is nothing but the Euclidean Fourier transform (up to the phase factor), and it is the Hankel transform for the minimal representation of the conformal group $O(N+1, 2)$ (see [35], see also [37] and [38, Introduction] for some perspectives of this direction in a more general setting).

A part of the results here has been announced in [3] without proof.

Notation $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, and $\mathbb{R}_{\geq 0} = \{t \in \mathbb{R} : t \geq 0\}$.

2. Preliminary results on Dunkl operators

2.1. Dunkl operators.

Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean scalar product in $\mathbb{R}^N$. We shall use the same notation for its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, denote by $||x|| = \langle x, x \rangle^{1/2}$.

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we write r_α for the reflection with respect to the hyperplane $\langle \alpha \rangle^\perp$ orthogonal to α defined by

$$r_\alpha(x) := x - 2\frac{\langle \alpha, x \rangle}{||\alpha||^2}\alpha, \qquad x \in \mathbb{R}^N.$$

We say a finite set $\mathscr{R}$ in $\mathbb{R}^N \setminus \{0\}$ is a (reduced) *root system* if:

(R1) $r_\alpha(\mathscr{R}) = \mathscr{R}$ for all $\alpha \in \mathscr{R}$,
(R2) $\mathscr{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \mathscr{R}$.

In this article, we do not impose crystallographic conditions on the roots, and do not require that $\mathscr{R}$ spans $\mathbb{R}^N$. However, we shall assume $\mathscr{R}$ is reduced, namely, (R2) is satisfied.

The subgroup $\mathfrak{C} \subset O(N, \mathbb{R})$ generated by the reflections $\{r_\alpha \mid \alpha \in \mathscr{R}\}$ is called the finite Coxeter group associated with $\mathscr{R}$. The Weyl groups such as the symmetric group $\mathfrak{S}_N$ for the type A_{N-1} root system and the hyperoctahedral group for the type B_N root system are typical examples. In addition, H_3, H_4 (icosahedral groups) and $I_2(n)$ (symmetry group of the regular n-gon) are also the Coxeter groups. We refer to [25] for more details on the theory of Coxeter groups.

Definition 2.1. *A multiplicity function for $\mathfrak{C}$ is a function $k : \mathscr{R} \to \mathbb{C}$ which is constant on $\mathfrak{C}$-orbits.*

Setting $k_\alpha := k(\alpha)$ for $\alpha \in \mathscr{R}$, we have $k_{h\alpha} = k_\alpha$ for all $h \in \mathfrak{C}$ from definition. We say k is non-negative if $k_\alpha \geq 0$ for all $\alpha \in \mathscr{R}$. The $\mathbb{C}$-vector space of multiplicity functions on $\mathscr{R}$ is denoted by $\mathscr{K}$. The dimension of $\mathscr{K}$ is equal to the number of $\mathfrak{C}$-orbits in $\mathscr{R}$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathscr{K}$, Dunkl [9] introduced a family of first order differential-difference operators $T_\xi(k)$ (*Dunkl's operators*) by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}, \qquad f \in C^1(\mathbb{R}^N). \tag{2.1}$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. Thanks to the $\mathfrak{C}$-invariance of the multiplicity function, this definition is independent of the choice of the positive subsystem $\mathscr{R}^+$. The operators $T_\xi(k)$ are homogeneous of degree -1. Moreover, the Dunkl operators satisfy the following properties (see [9]):

(D1) $L(h) \circ T_\xi(k) \circ L(h)^{-1} = T_{h\xi}(k)$ for all $h \in \mathfrak{C}$,
(D2) $T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k)$ for all $\xi, \eta \in \mathbb{R}^N$,
(D3) $T_\xi(k)[fg] = gT_\xi(k)f + fT_\xi(k)g$ if f and g are in $C^1(\mathbb{R}^N)$ and at least one of them is $\mathfrak{C}$-invariant.

Here, we denote by $L(h)$ the left regular action of $h \in \mathfrak{C}$ on the function space on $\mathbb{R}^N$:

$$(L(h)f)(x) := f(h^{-1} \cdot x).$$

Remark 2.2. *The Dunkl Laplacian arises as the radial part of the Laplacian on the tangent space of a Riemannian symmetric spaces. Let* $\mathfrak{g}$ *be a real semisimple Lie algebra with Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. *We take a maximal abelian subspace* $\mathfrak{a}$ *in* $\mathfrak{p}$, *and let* $\Sigma(\mathfrak{g}, \mathfrak{a})$ *be the set of restricted roots, and* m_α *the multiplicity of* $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. *We may consider* $\Sigma(\mathfrak{g}, \mathfrak{a})$ *to be a subset of* $\mathfrak{a}$ *by means of the Killing form of* $\mathfrak{g}$. *The Killing form endows* $\mathfrak{p}$ *with a flat Riemannian symmetric space structure, and we write* $\Delta_\mathfrak{p}$ *for the (Euclidean) Laplacian on* $\mathfrak{p}$. *Put* $\mathscr{R} := 2\Sigma(\mathfrak{g}, \mathfrak{a})$ *and* $k_\alpha := \frac{1}{2}\sum_{\beta\in\Sigma^+\cap\mathbb{R}\alpha} m_\beta$. *We note that the root system* $\mathscr{R}$ *is not necessarily reduced. Then the radial part of* $\Delta_\mathfrak{p}$, *denoted by* $Rad(\Delta_\mathfrak{p})$, *(see* [24, Proposition 3.13]*) is given by*

$$Rad(\Delta_\mathfrak{p})f = \Delta_k f$$

for every $\mathfrak{C}$*-invariant function* $f \in C^\infty(\mathfrak{a})$, *where* Δ_k *is the Dunkl Laplacian which will be defined in* (2.10)*.*

Remark 2.3. *Some of our results still hold for "slightly-negative" multiplicity functions. For instance, when* $k_\alpha = k$ *for all* $\alpha \in \mathscr{R}$, *we may relax the assumption* $k \geq 0$ *by* $k > -\frac{1}{d_{\max}}$ *where* $d_{\max}$ *is the largest fundamental degree of the Coxeter group* $\mathfrak{C}$ *(see* [16, Theorem 3.1]*). However, for simplicity, we will restrict ourselves to non-negative multiplicity functions* $k = (k_\alpha)_{\alpha\in\mathscr{R}}$.

Let ϑ_k be the weight function on $\mathbb{R}^N$ defined by

$$\vartheta_k(x) := \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, x\rangle|^{2k_\alpha}, \qquad x \in \mathbb{R}^N. \tag{2.2}$$

It is $\mathfrak{C}$-invariant and homogeneous of degree $2\langle k\rangle$, where the index $\langle k\rangle$ of the multiplicity function k is defined as

$$\langle k\rangle := \sum_{\alpha\in\mathscr{R}^+} k_\alpha = \frac{1}{2}\sum_{\alpha\in\mathscr{R}} k_\alpha. \tag{2.3}$$

Let dx be the Lebesgue measure on $\mathbb{R}^N$ with respect to the inner product $\langle\ ,\ \rangle$. Then the Dunkl operators are skew-symmetric with respect to the measure $\vartheta_k(x)dx$ (see [9]). In particular, if f and g are differentiable and one of them has compact support, then

$$\int_{\mathbb{R}^N} (T_\xi(k)f)(x)g(x)\vartheta_k(x)dx = -\int_{\mathbb{R}^N} f(x)(T_\xi(k)g)(x)\vartheta_k(x)dx. \tag{2.4}$$

It is shown in [10] that for any non-negative root multiplicity function k there is a unique linear isomorphism V_k (*Dunkl's intertwining operator*) on the space $\mathscr{P}(\mathbb{R}^N)$ of polynomial functions on $\mathbb{R}^N$ such that:

(I1) $V_k(\mathscr{P}_m(\mathbb{R}^N)) = \mathscr{P}_m(\mathbb{R}^N)$ for all $m \in \mathbb{N}$,
(I2) $V_{k|\mathscr{P}_0(\mathbb{R}^N)} = \mathrm{id}$,
(I3) $T_\xi(k)V_k = V_k\partial_\xi$ for all $\xi \in \mathbb{R}^N$.

Here, $\mathscr{P}_m(\mathbb{R}^N)$ denotes the space of homogeneous polynomials of degree m. It is known that V_k induces a homeomorphism of $C(\mathbb{R}^N)$ and also that of $C^\infty(\mathbb{R}^N)$ (cf. [56]). See also [15] for more results on V_k for $\mathbb{C}$-valued multiplicity functions on $\mathscr{R}$.

For arbitrary finite reflection group $\mathfrak{C}$, and for any non-negative multiplicity function k, Rösler [49] proved that there exists a unique positive Radon probability-measure μ_x^k on $\mathbb{R}^N$ such that

$$V_k f(x) = \int_{\mathbb{R}^N} f(\xi)d\mu_x^k(\xi). \tag{2.5}$$

The measure μ_x^k depends on $x \in \mathbb{R}^N$ and its support is contained in the ball $B(\|x\|) := \{\xi \in \mathbb{R}^N \mid \|\xi\| \le \|x\|\}$. Moreover, for any Borel set $S \subset \mathbb{R}^N$, $g \in \mathfrak{C}$ and $r > 0$, the following invariant property holds:

$$\mu_x^k(S) = \mu_{gx}^k(gS) = \mu_{rx}^k(rS).$$

In view of the Laplace type representation (2.5), Dunkl's intertwining operator V_k can be extended to a larger class of spaces. For example, let B denote the closed unit ball in $\mathbb{R}^N$. Then the support property of μ_x^k leads us to the following:

Lemma 2.4. *For any $R > 0$, V_k induces a continuous endomorphism of $C(B(R))$.*

Proof. Let $f \in C(B(R))$. We extend f to be a continuous function $\widetilde{f}$ on $\mathbb{R}^N$. Then, $V_k\widetilde{f}$ is given by the integral

$$V_k\widetilde{f}(x) = \int_{\mathbb{R}^N} \widetilde{f}(\xi)d\mu_x^k(\xi).$$

Suppose now $x \in B(R)$. Then $\operatorname{Supp}\mu_x^k \subset B(\|x\|) \subset B(R)$. Hence, $(V_k\tilde{f})|_{B(R)}$ is determined by the restriction $f = \tilde{f}|_{B(R)}$. Thus, the correspondence $f \mapsto (V_k\tilde{f})|_{B(R)}$ is well-defined, and we get an induced linear map $V_k \colon C(B(R)) \to C(B(R))$, by using the same letter.

Next, suppose a sequence $f_j \in C(B(R))$ converges uniformly to $f \in C(B(R))$ as $j \to \infty$. Then we can extend f_j to a continuous function $\widetilde{f_j}$ on $\mathbb{R}^N$ such that $\widetilde{f_j}$ converges to $\widetilde{f}$ on every compact set on $\mathbb{R}^N$. Hence $V_k\widetilde{f_j}$ converges to $V_k\widetilde{f}$, and so does V_kf_j to V_kf. □

For a continuous function $h(t)$ of one variable, we set

$$h_y(\cdot) := h(\langle\cdot, y\rangle) \quad (y \in \mathbb{R}^N),$$

and define

$$(\widetilde{V}_k h)(x, y) := (V_k h_y)(x) = \int_{\mathbb{R}^N} h(\langle\xi, y\rangle)d\mu_x^k(\xi). \tag{2.6}$$

Then, $(\widetilde{V}_k h)(x, y)$ is a continuous function on $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

We note that if $k \equiv 0$ then

$$(\widetilde{V}_0 h)(x, y) = h(\langle x, y\rangle).$$

If $h(t)$ is defined only near the origin, we can still get a continuous function $(\widetilde{V_k}h)(x,y)$ as far as $|\langle x,y\rangle|$ is sufficiently small. To be more precise, we prepare the following proposition for later purpose. For simplicity, we write B for the unit ball $B(1)$ in $\mathbb{R}^N$.

Proposition 2.5. *Suppose $h(t)$ is a continuous function on the closed interval $[-1,1]$. Then, $(\widetilde{V_k}h)(x,y)$ is a continuous function on $B\times B$. Further, $\widetilde{V_k}h$ satisfies*

$$\|\widetilde{V_k}h\|_{L^\infty(B\times B)} \le \|h\|_{L^\infty([-1,1])} \tag{2.7}$$

$$(\widetilde{V_k}h)(x,y) = (\widetilde{V_k}h)(y,x). \tag{2.8}$$

Proof. We extend h to a continuous function $\widetilde{h}$ on $\mathbb{R}$. It follows from Lemma 2.4 that the values $(\widetilde{V_k}\widetilde{h})(x,y)$ for (x,y) satisfying $|\langle x,y\rangle|\le 1$ are determined by the restriction $h=\widetilde{h}|_{[-1,1]}$. Hence,

$$(\widetilde{V_k}h)(x,y) := (\widetilde{V_k}\widetilde{h})(x,y), \quad (x,y)\in B\times B$$

is well-defined.

Since μ_x^k is a probability measure, we get an upper estimate (2.7) from the integral expression (2.6).

By the Weierstrass theorem, we can find a sequence of polynomials $h_j(t)$ $(j=1,2,\dots)$ such that $h_j(t)$ converges to $\widetilde{h}(t)$ uniformly on any compact set of $\mathbb{R}$. Then, $\widetilde{V_k}h_j$ converges to $\widetilde{V_k}h$ uniformly on $B\times B$. Thanks to [10, Proposition 3.2], we have $(\widetilde{V_k}h_j)(x,y)=(\widetilde{V_k}h_j)(y,x)$. Taking the limit as j tends to infinity, we get the equation (2.8). Hence, Proposition 2.5 is proved. □

Aside from the development of the general theory of the Dunkl transform, we note that explicit formulas for V_k have been known for only a few cases: $\mathfrak{C}=\mathbb{Z}_2^N$, $\mathfrak{C}=S_3$, and the equal parameter case for the Weyl group of B_2 (see [13] for the recent survey by C. Dunkl).

2.2. **The Dunkl Laplacian.**

Let $\{\xi_1,\dots,\xi_N\}$ be an orthonormal basis of $(\mathbb{R}^N,\langle\cdot,\cdot\rangle)$. For the j-th basis vector ξ_j, we will use the abbreviation $T_{\xi_j}(k)=T_j(k)$. The Dunkl–Laplace operator, or simply, the Dunkl Laplacian, is defined as

$$\Delta_k := \sum_{j=1}^N T_j(k)^2. \tag{2.9}$$

The definition of Δ_k is independent of the choice of an orthonormal basis of $\mathbb{R}^N$. In fact, it is proved in [9] that Δ_k is expressed as

$$\Delta_k f(x) = \Delta f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \left\{ \frac{2\langle\nabla f(x),\alpha\rangle}{\langle\alpha,x\rangle} - \|\alpha\|^2 \frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle^2} \right\}, \tag{2.10}$$

where ∇ denotes the usual gradient operator.

For $k\equiv 0$, the Dunkl–Laplace operator Δ_k reduces to the Euclidean Laplacian Δ, which commutes with the action of $O(N)$. For general k, it follows from (D1) and (2.9) that Δ_k commutes with the action of the Coxeter group $\mathfrak{C}$, i.e.

$$L(h)\circ\Delta_k\circ L(h)^{-1} = \Delta_k, \qquad \forall h\in\mathfrak{C}. \tag{2.11}$$

Definition 2.6. *A k-harmonic polynomial of degree m ($m \in \mathbb{N}$) is a homogeneous polynomial p on $\mathbb{R}^N$ of degree m such that $\Delta_k p = 0$.*

Denote by $\mathscr{H}_k^m(\mathbb{R}^N)$ the space of k-harmonic polynomials of degree m. It is naturally a representation space of the Coxeter group $\mathfrak{C}$.

Let $d\sigma$ be the standard measure on S^{N-1}, ϑ_k the density given in (2.2), and d_k the normalizing constant defined by

$$d_k := \Big(\int_{S^{N-1}} \vartheta_k(\omega) d\sigma(\omega) \Big)^{-1}. \tag{2.12}$$

We write $L^2(S^{N-1}, \vartheta_k(\omega)d\sigma(\omega))$ for the Hilbert space with the following inner product $\langle\ ,\ \rangle_k$ given by

$$\langle f, g\rangle_k := d_k \int_{S^{N-1}} f(\omega)\overline{g(\omega)}\vartheta_k(\omega)d\sigma(\omega).$$

For $k \equiv 0$, d_k^{-1} is the volume of the unit sphere, namely,

$$d_0 = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}. \tag{2.13}$$

Thanks to Selberg, Mehta, Macdonald [42], Heckman, Opdam [45], and others, there is a closed form of d_k in terms of Gamma functions when k is a non-negative multiplicity function (see also [16]).

As in the classical spherical harmonics (i.e. the $k \equiv 0$ case), we have (see [8, page 37]):

Fact 2.7.

1) $\mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}$ ($m = 0, 1, 2, \dots$) *are orthogonal to each other with respect to* $\langle\ ,\ \rangle_k$.
2) *The Hilbert space $L^2(S^{N-1}, \vartheta_k(\omega)d\sigma(\omega))$ decomposes as a direct Hilbert sum:*

$$L^2(S^{N-1}, \vartheta_k(\omega)d\sigma(\omega)) = \sum_{m\in\mathbb{N}}^{\oplus} \mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}. \tag{2.14}$$

We pin down some basic formulae of Δ_k. We write the Euler operator as

$$E := \sum_{j=1}^{N} x_j \partial_j. \tag{2.15}$$

Lemma 2.8. 1) *The Dunkl Laplacian Δ_k is of degree -2, namely,*

$$[E, \Delta_k] = -2\Delta_k \tag{2.16}$$

2)

$$\sum_{j=1}^{N} (x_j T_j(k) + T_j(k) x_j) = N + 2\langle k\rangle + 2E. \tag{2.17}$$

3) *Suppose $\psi(r)$ is a C^∞ function of one variable. Then we have*

$$[\Delta_k, \psi(\|x\|^a)] = a^2\|x\|^{2a-2}\psi''(\|x\|^a) + a\|x\|^{a-2}\psi'(\|x\|^a)((N + 2\langle k\rangle + a - 2) + 2E). \tag{2.18}$$

Proof. See [23, Theorem 3.3] for 1) and 2).

3) Take an arbitrary C^∞ function f on $\mathbb{R}^N$. We recall from the definition (2.1) and (D3) that

$$\begin{aligned} T_j(k)g &= \partial_j g, \\ T_j(k)(fg) &= (T_j(k)f)g + f(\partial_j g), \end{aligned} \tag{2.19}$$

if g is a $\mathfrak{C}$-invariant function on $\mathbb{R}^N$. In particular,

$$T_j(k)\psi(\|x\|^a) = ax_j\|x\|^{a-2}\psi'(\|x\|^a),$$

$$T_j(k)(f(x)\psi(\|x\|^a)) = (T_j(k)f(x))\psi(\|x\|^a) + ax_j f(x)\|x\|^{a-2}\psi'(\|x\|^a).$$

Using (D3) again, we get

$$\begin{aligned} T_j(k)^2(f(x)\psi(\|x\|^a)) = (T_j(k)^2 f(x))\psi(\|x\|^a) \\ + a\|x\|^{a-2}\psi'(\|x\|^a)(x_j(T_j(k)f(x)) + T_j(k)(x_j f(x))) \\ + af(x)x_jT_j(k)(\|x\|^{a-2}\psi'(\|x\|^a)). \end{aligned}$$

Taking the summation over j, we arrive at

$$\begin{aligned} \Delta_k(f(x)\psi(\|x\|^a)) = (\Delta_k f(x))\psi(\|x\|^a) + a\|x\|^{a-2}\psi'(\|x\|^a)(2E + N + 2\langle k\rangle)f(x) \\ + af(x)E(\|x\|^{a-2}\psi'(\|x\|^a)). \end{aligned}$$

Here, we have used the expression (2.9) of Δ_k, (2.17), and (2.19). Now, (2.18) follows from the following observation: in the polar coordinate $x = r\omega$, the Euler operator E amounts to $r\frac{\partial}{\partial r}$, and $r\frac{d}{dr}(r^{a-2}\psi'(r^a)) = (a-2)r^{a-2}\psi'(r^a) + ar^{2a-2}\psi''(r^a)$. □

To end this section, we consider a '(k, a)-deformation' of the classical formula

$$e^{\|x\|^2} \circ \Delta \circ e^{-\|x\|^2} = \Delta + 4\|x\|^2 - 2N - 4E.$$

Lemma 2.9. *For any $\nu \in \mathbb{C}$ and $a \neq 0$, we have*

$$e^{\frac{\nu}{a}\|x\|^a} \circ \|x\|^{2-a}\Delta_k \circ e^{-\frac{\nu}{a}\|x\|^a} = \|x\|^{2-a}\Delta_k + \nu^2\|x\|^a - \nu((N + 2\langle k\rangle + a - 2) + 2E). \tag{2.20}$$

Proof. The proof parallels to that of Lemma 2.8 3). By the property (D3) of the Dunkl operators, we get

$$T_j(k)(e^{\lambda\|x\|^a}h(x)) = (T_j(k)e^{\lambda\|x\|^a})h(x) + e^{\lambda\|x\|^a}T_j(k)h(x).$$

Then, substituting the formula

$$T_j(k)e^{\lambda\|x\|^a} = \partial_j e^{\lambda\|x\|^a} = \lambda ax_j\|x\|^{a-2}e^{\lambda\|x\|^a},$$

we have

$$e^{-\lambda\|x\|^a} \circ T_j(k) \circ e^{\lambda\|x\|^a}h(x) = \lambda ax_j\|x\|^{a-2}h(x) + T_j(k)h(x). \tag{2.21}$$

Iterating (2.21) and using

$$T_j(k)\|x\|^{a-2} = (a-2)x_j\|x\|^{a-4},$$

we get

$$\begin{aligned} e^{-\lambda\|x\|^a} \circ T_j(k)^2 \circ e^{\lambda\|x\|^a} &= (\lambda ax_j\|x\|^{a-2} + T_j(k))^2 \\ &= \lambda^2a^2x_j^2\|x\|^{2a-2} + \lambda a\|x\|^{a-2}(x_jT_j(k) + T_j(k)x_j) \\ &\quad + \lambda a(a-2)x_j^2\|x\|^{a-4} + T_j(k)^2. \end{aligned}$$

Summing them up over j, we have

$$e^{-\lambda\|x\|^a} \circ \Delta_k \circ e^{\lambda\|x\|^a} = \Delta_k + \lambda^2 a^2 \|x\|^{2a} + \lambda a \|x\|^{a-2}(a - 2 + \sum_{j=1}^{N}(x_j T_j(k) + T_j(k)x_j)) \tag{2.22}$$

The substitution of (2.17) and $\lambda = -\frac{\nu}{a}$ to (2.22) shows Lemma. □

3. The infinitesimal representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$

3.1. $\mathfrak{sl}_2$ triple of differential-difference operators.

In this subsection, we construct a family of Lie algebras which are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ in the space of differential-difference operators on $\mathbb{R}^N$. This family is parametrized by a non-zero complex number a and a multiplicity function k for the Coxeter group.

We take a basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as

$$\mathbf{e}^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

The triple $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{h}\}$ satisfies the commutation relations

$$[\mathbf{e}^+, \mathbf{e}^-] = \mathbf{h}, \qquad [\mathbf{h}, \mathbf{e}^+] = 2\mathbf{e}^+, \qquad [\mathbf{h}, \mathbf{e}^-] = -2\mathbf{e}^-. \tag{3.2}$$

Definition 3.1. *An $\mathfrak{sl}_2$ triple is a triple of non-zero elements in a Lie algebra satisfying the same relation with* (3.2)*.*

We recall from Section 2 that Δ_k is the Dunkl Laplacian associated with a multiplicity function k on the root system, and that $\langle k \rangle$ is the index defined in (2.3). For a non-zero complex parameter a, we introduce the following differential-difference operators on $\mathbb{R}^N$:

$$\mathbb{E}^+_{k,a} := \frac{i}{a}\|x\|^a, \qquad \mathbb{E}^-_{k,a} := \frac{i}{a}\|x\|^{2-a}\Delta_k, \qquad \mathbb{H}_{k,a} := \frac{N + 2\langle k \rangle + a - 2}{a} + \frac{2}{a}\sum_{i=1}^{N} x_i \partial_i. \tag{3.3}$$

The point of the definition is:

Theorem 3.2. *The operators $\mathbb{E}^+_{k,a}$, $\mathbb{E}^-_{k,a}$ and $\mathbb{H}_{k,a}$ form an $\mathfrak{sl}_2$ triple for any complex number $a \neq 0$ and any multiplicity function k.*

Proof of Theorem 3.2. The operator $\mathbb{E}^+_{k,a}$ is homogeneous of degree a, and $\mathbb{E}^-_{k,a}$ is of degree $(2 - a) - 2 = -a$ by Lemma 2.8 1). Let $E = \sum_{j=1}^{N} x_j \partial_j$ be the Euler operator as in (2.15). Since $\mathbb{H}_{k,a}$ is of the form $\frac{2}{a}E$ + constant, the identity $[\mathbb{H}_{k,a}, \mathbb{E}^\pm_{k,a}] = \pm 2\mathbb{E}^\pm_{k,a}$ is now clear.

To see $[\mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}] = \mathbb{H}_{k,a}$, we apply Lemma 2.8 3) to the function $\psi(r) = r$. Then we get

$$\Delta_k \circ \|x\|^a - \|x\|^a \Delta_k = a(N + 2\langle k \rangle + a - 2)\|x\|^{a-2} + 2a\|x\|^{a-2}E. \tag{3.4}$$

Composing the multiplication operator $\|x\|^{2-a}$, we have

$$\|x\|^{2-a}\Delta_k \circ \|x\|^a - \|x\|^2\Delta_k = a(N + 2\langle k \rangle + a - 2) + 2aE.$$

In view of the definition (3.3), this means $[\mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}] = \mathbb{H}_{k,a}$.

Hence, Theorem 3.2 is proved. □

Remark 3.3. *Theorem 3.2 for particular cases was previously known.*

(1) *For* $a = 2$ *and* $k \equiv 0$, $\{\mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}, \mathbb{H}_{k,a}\}$ *is the classical harmonic* $\mathfrak{sl}_2$ *triple* $\{\frac{i}{2}\|x\|^2, \frac{i}{2}\Delta, \frac{N}{2} + \sum_i x_i\partial_i\}$. *This* $\mathfrak{sl}_2$ *triple was used in the analysis of the Schrödinger model of the Weil representation of the metaplectic group* $Mp(N,\mathbb{R})$ *(see Howe* [29], *Howe–Tan* [30]*).*
(2) *For* $a = 2$ *and* $k > 0$, *Theorem 3.2 was proved in Heckman* [23, Theorem 3.3].
(3) *For* $a = 1$ *and* $k \equiv 0$, $\{\mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}, \mathbb{H}_{k,a}\}$ *is the* $\mathfrak{sl}_2$ *triple introduced in Kobayashi and Mano* [35, 36] *where the authors studied the* L^2*-model of the minimal representation of the double covering group of* $SO_0(N+1,2)$. *(To be more precise, the formulas in* [36] *are given for the* $\mathfrak{sl}_2$ *triple for* $\{2\mathbb{E}^+_{k,a}, \frac{1}{2}\mathbb{E}^-_{k,a}, \mathbb{H}_{k,a}\}$ *in our notation.)*
(4) *For* $k \equiv 0$, *the deformation parameter* a *was also considered in Mano* [43].

The differential-difference operators (3.3) stabilize $C^\infty(\mathbb{R}^N \setminus \{0\})$, the space of (complex valued) smooth functions on $\mathbb{R}^N \setminus \{0\}$. Thus, for each non-zero complex number a and each multiplicity function k on the root system, we can define an $\mathbb{R}$-linear map

$$\omega_{k,a} : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{End}(C^\infty(\mathbb{R}^N \setminus \{0\})) \tag{3.5}$$

by setting

$$\omega_{k,a}(\mathbf{h}) = \mathbb{H}_{k,a}, \qquad \omega_{k,a}(\mathbf{e}^+) = \mathbb{E}^+_{k,a}, \qquad \omega_{k,a}(\mathbf{e}^-) = \mathbb{E}^-_{k,a}. \tag{3.6}$$

Then, Theorem 3.2 implies that $\omega_{k,a}$ is a Lie algebra homomorphism.

We denote by $U(\mathfrak{sl}(2,\mathbb{C}))$ the universal enveloping algebra of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$. Then, we can extend (3.5) to a $\mathbb{C}$-algebra homomorphism (by the same symbol)

$$\omega_{k,a} : U(\mathfrak{sl}(2,\mathbb{C})) \to \mathrm{End}(C^\infty(\mathbb{R}^N \setminus \{0\})).$$

We use the letter L to denote by the left regular representation of the Coxeter group $\mathfrak{C}$ on $C^\infty(\mathbb{R}^N \setminus \{0\})$.

Lemma 3.4. *The two actions* L *of the Coxeter group* $\mathfrak{C}$ *and* $\omega_{k,a}$ *of the Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$ *commute.*

Proof. Obviously, $L(h)$ commutes with the multiplication operator $\mathbb{E}^+_{k,a} = \frac{i}{a}\|x\|^a$. As we saw in (2.11), $L(h)$ commutes with the Dunkl Laplacian. Hence, it commutes also with $\mathbb{E}^-_{k,a}$. Finally, the commutation relation $[\mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}] = \mathbb{H}_{k,a}$ implies $L(h) \circ \mathbb{H}_{k,a} = \mathbb{H}_{k,a} \circ L(h)$. □

We consider the following unitary matrix

$$c := \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}. \tag{3.7}$$

We set

$$\mathfrak{su}(1,1) := \{X \in \mathfrak{sl}(2,\mathbb{C}) : X^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X = 0\},$$

another real form of $\mathfrak{sl}(2,\mathbb{C})$. Then, $\mathrm{Ad}(c)$ induces a Lie algebra isomorphism (the Cayley transform)

$$\mathrm{Ad}(c) : \mathfrak{sl}(2,\mathbb{R}) \xrightarrow{\sim} \mathfrak{su}(1,1).$$

We set

$$\mathbf{k} := \mathrm{Ad}(c)\mathbf{h} = i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \frac{1}{i}(\mathbf{e}^+ - \mathbf{e}^-), \tag{3.8 a}$$

$$\mathbf{n}^+ := \mathrm{Ad}(c)\mathbf{e}^+ = \frac{1}{2}\begin{pmatrix}i & -1\\ -1 & -i\end{pmatrix} = \frac{1}{2i}(-\mathbf{h} + \frac{1}{i}\mathbf{e}^+ + \frac{1}{i}\mathbf{e}^-), \tag{3.8 b}$$

$$\mathbf{n}^- := \mathrm{Ad}(c)\mathbf{e}^- = \frac{1}{2}\begin{pmatrix}-i & -1\\ -1 & i\end{pmatrix} = \frac{1}{2i}(\mathbf{h} + \frac{1}{i}\mathbf{e}^+ + \frac{1}{i}\mathbf{e}^-). \tag{3.8 c}$$

Correspondingly to (3.8 a – c), the Cayley transform of the operators (3.6) amounts to:

$$\widetilde{\mathbb{H}}_{k,a} := \omega_{k,a}(\mathbf{k}) = \frac{\|x\|^a - \|x\|^{2-a}\Delta_k}{a} = -\frac{1}{a}\Delta_{k,a}, \tag{3.9 a}$$

$$\widetilde{\mathbb{E}}^+_{k,a} := \omega_{k,a}(\mathbf{n}^+) = i\frac{2E + (N + 2\langle k\rangle + a - 2) - \|x\|^{2-a}\Delta_k - \|x\|^a}{2a}, \tag{3.9 b}$$

$$\widetilde{\mathbb{E}}^-_{k,a} := \omega_{k,a}(\mathbf{n}^-) = -i\frac{2E + (N + 2\langle k\rangle + a - 2) + \|x\|^{2-a}\Delta_k + \|x\|^a}{2a}. \tag{3.9 c}$$

Here, $E = \sum_{i=1}^N x_i\partial_i$ is the Euler operator.

Since $\mathrm{Ad}(c)$ gives a Lie algebra isomorphism, $\{\widetilde{\mathbb{E}}^+_{k,a}, \widetilde{\mathbb{E}}^-_{k,a}, \widetilde{\mathbb{H}}_{k,a}\}$ also forms an $\mathfrak{sl}_2$ triple of differential-difference operators. Putting $\nu = \pm 1$ in Lemma 2.9, we get another expression of the triple $\{\widetilde{\mathbb{E}}^+_{k,a}, \widetilde{\mathbb{E}}^-_{k,a}, \widetilde{\mathbb{H}}_{k,a}\}$ as follows:

Lemma 3.5. *Let $\widetilde{\mathbb{E}}^+_{k,a}$, $\widetilde{\mathbb{E}}^-_{k,a}$, and $\widetilde{\mathbb{H}}_{k,a}$ be as in* (3.9 a, b, c)*. Then, we have:*

$$\widetilde{\mathbb{E}}^+_{k,a} = \omega_{k,a}(\mathbf{n}^+) = -\frac{i}{2a}e^{\frac{\|x\|^a}{a}} \circ \|x\|^{2-a}\Delta_k \circ e^{-\frac{\|x\|^a}{a}}, \tag{3.10 a}$$

$$\widetilde{\mathbb{E}}^-_{k,a} = \omega_{k,a}(\mathbf{n}^-) = -\frac{i}{2a}e^{-\frac{\|x\|^a}{a}} \circ \|x\|^{2-a}\Delta_k \circ e^{\frac{\|x\|^a}{a}}, \tag{3.10 b}$$

$$\begin{aligned}\widetilde{\mathbb{H}}_{k,a} = \omega_{k,a}(\mathbf{k}) &= e^{-\frac{\|x\|^a}{a}} \circ \left(\mathbb{H}_{k,a} - \frac{\|x\|^{2-a}\Delta_k}{a}\right) \circ e^{\frac{\|x\|^a}{a}} \\ &= \frac{1}{a}e^{-\frac{\|x\|^a}{a}} \circ \Big((N + 2\langle k\rangle + a - 2) + 2E - \|x\|^{2-a}\Delta_k\Big) \circ e^{\frac{\|x\|^a}{a}}.\end{aligned} \tag{3.10 c}$$

3.2. **Differential-difference operators in the polar coordinate.**

In this subsection, we rewrite the differential-difference operators introduced in Section 3.1 by means of the polar coordinate.

We set

$$\lambda_{k,a,m} := \frac{2m + 2\langle k\rangle + N - 2}{a}. \tag{3.11}$$

We begin with the following lemma.

Lemma 3.6. *Retain the notation of Section 2.2. For all $\psi \in C^\infty(\mathbb{R}_+)$ and $p \in \mathscr{H}^m_k(\mathbb{R}^N)$, we have*

$$\mathbb{H}_{k,a}\Big(p(x)\psi(\|x\|^a)\Big) = \Big\{(\lambda_{k,a,m} + 1)\psi(\|x\|^a) + 2\|x\|^a\psi'(\|x\|^a)\Big\}p(x), \tag{3.12}$$

$$\Delta_k\Big(p(x)\psi(\|x\|^a)\Big) = \Big\{a^2(\lambda_{k,a,m} + 1)\|x\|^{a-2}\psi'(\|x\|^a) + a^2\|x\|^{2a-2}\psi''(\|x\|^a)\Big\}p(x). \tag{3.13}$$

Proof. The first statement is straightforward because the Euler operator E is of the form $r\frac{\partial}{\partial r}$ in the polar coordinates $x = r\omega$. To see the second statement, we apply (2.18) to $p(x)$. Since $Ep = mp$ and $\Delta_k p = 0$, we get the desired formula (3.13). $\square$

We consider the following linear operator:

$$T_a : C^\infty(\mathbb{R}^N) \otimes C^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}^N \setminus \{0\}), \quad (p, \psi) \mapsto p(x)\psi(\|x\|^a) \tag{3.14}$$

Lemma 3.7. *Via the linear map T_a, the operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$, and $\mathbb{E}^-_{k,a}$ (see* (3.3)*) take the following forms on $\mathscr{H}^m_k(\mathbb{R}^N) \otimes C^\infty(\mathbb{R}_+)$:*

$$\mathbb{H}_{k,a} \circ T_a = T_a \circ \Big(\mathrm{id} \otimes \Big(2r\frac{d}{dr} + (\lambda_{k,a,m} + 1)\Big)\Big) \tag{3.15 a}$$

$$\mathbb{E}^+_{k,a} \circ T_a = T_a \circ \Big(\mathrm{id} \otimes \frac{i}{a} r\Big) \tag{3.15 b}$$

$$\mathbb{E}^-_{k,a} \circ T_a = T_a \circ \Big(\mathrm{id} \otimes ai\Big(r\frac{d^2}{dr^2} + (\lambda_{k,a,m} + 1)\frac{d}{dr}\Big)\Big) \tag{3.15 c}$$

Proof. Clear from Lemma 3.6 and the definition (3.3) of $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$, and $\mathbb{E}^-_{k,a}$. $\square$

The point of Lemma 3.7 is that the operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$, and $\mathbb{E}^-_{k,a}$ act only on the radial part ψ when applied to those functions $p(x)\psi(\|x\|^a)$ for $p \in \mathscr{H}^m_k(\mathbb{R}^N)$.

For $a > 0$, we define an endomorphism of $C^\infty(\mathbb{R}_+)$ by

$$U_a : C^\infty(\mathbb{R}_+) \xrightarrow{\sim} C^\infty(\mathbb{R}_+), \; g(t) \mapsto (U_a g)(r) := \exp\Big(-\frac{1}{a}r\Big)g\Big(\frac{2}{a}r\Big).$$

Clearly, U_a is invertible. Composing with T_a (see (3.14)), we define the following linear operator S_a by

$$S_a := T_a \circ (\mathrm{id} \otimes U_a).$$

That is, $S_a : C^\infty(\mathbb{R}^N) \otimes C^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}^N \setminus \{0\})$ is given by

$$S_a(p \otimes g)(x) := p(x) \exp\Big(-\frac{1}{a}\|x\|^a\Big)g\Big(\frac{2}{a}\|x\|^a\Big). \tag{3.16}$$

We set

$$P_{t,\lambda} := t\frac{d^2}{dt^2} + (\lambda_{k,a,m} + 1 - t)\frac{d}{dt}. \tag{3.17}$$

Here, λ stands for $\lambda_{k,a,m}$. Then Lemma 3.7 can be formulated as follows:

Lemma 3.8. *Via the map S_a, the operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$, and $\mathbb{E}^-_{k,a}$ take the following forms on $\mathscr{H}^m_k(\mathbb{R}^N) \otimes C^\infty(\mathbb{R}_+)$:*

$$\mathbb{H}_{k,a} \circ S_a = S_a \circ \Big(\mathrm{id} \otimes \Big(2t\frac{d}{dt} + (\lambda_{k,a,m} + 1 - t)\Big)\Big),$$

$$\mathbb{E}^+_{k,a} \circ S_a = S_a \circ \Big(\mathrm{id} \otimes \frac{i}{2}t\Big),$$

$$\mathbb{E}^-_{k,a} \circ S_a = S_a \circ \Big(\mathrm{id} \otimes i\Big(2P_{t,\lambda} + \frac{t}{2} - \lambda_{k,a,m} - 1\Big)\Big).$$

Proof. Immediate from Lemma 3.7 and the following relations:

$$\begin{aligned} U_a^{-1} \circ \frac{d}{dr} \circ U_a &= \frac{2}{a}\left(\frac{d}{dt} - \frac{1}{2}\right), \\ U_a^{-1} \circ r \circ U_a &= \frac{a}{2} t. \end{aligned}$$

□

Similarly, by using (3.8 a–c), the actions of $\widetilde{\mathbb{H}}_{k,a}$, $\widetilde{\mathbb{E}}^+_{k,a}$, and $\widetilde{\mathbb{E}}^-_{k,a}$ (see (3.9 a–c)) are given as follows:

Lemma 3.9. *Let $P_{t,\lambda}$ be as in* (3.17)*. Then, through the linear map S_a (see* (3.16)*), $\widetilde{\mathbb{H}}_{k,a}$, $\widetilde{\mathbb{E}}^+_{k,a}$, and $\widetilde{\mathbb{E}}^-_{k,a}$ take the following forms on $\mathscr{H}^m_k(\mathbb{R}^N) \otimes C^\infty(\mathbb{R}_+)$:*

$$\begin{aligned} \widetilde{\mathbb{H}}_{k,a} \circ S_a &= S_a \circ \Big(\mathrm{id} \otimes \big(-2P_{t,\lambda} + \lambda_{k,a,m} + 1\big)\Big), \\ \widetilde{\mathbb{E}}^+_{k,a} \circ S_a &= S_a \circ \Big(\mathrm{id} \otimes \Big(-i\Big(P_{t,\lambda} - t\frac{d}{dt} + t - \lambda_{k,a,m} - 1\Big)\Big)\Big), \\ \widetilde{\mathbb{E}}^-_{k,a} \circ S_a &= S_a \circ \Big(\mathrm{id} \otimes \Big(-i\Big(P_{t,\lambda} + t\frac{d}{dt}\Big)\Big)\Big). \end{aligned}$$

3.3. **Laguerre polynomials revisited.**

In this subsection, after a brief summary on the (classical) Laguerre polynomials we give a 'non-standard' representation of them in terms of the one parameter group with infinitesimal generator $t\frac{d^2}{dt^2} + (\lambda+1)\frac{d}{dt}$ (see Proposition 3.11).

For a complex number $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -1$, we write $L^{(\lambda)}_\ell$ for the Laguerre polynomial defined by

$$L^{(\lambda)}_\ell(t) := \frac{(\lambda+1)_\ell}{\ell!} \sum_{j=0}^{\ell} \frac{(-\ell)_j}{(\lambda+1)_j} \frac{t^j}{j!} = \sum_{j=0}^{\ell} \frac{(-1)^j \Gamma(\lambda+\ell+1)}{(\ell-j)!\Gamma(\lambda+j+1)} \frac{t^j}{j!}.$$

Here, $(a)_m := a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol.

We list some standard properties of Laguerre polynomials that we shall use in this article.

Fact 3.10 (see [1, §6.5])**.** *Suppose* $\operatorname{Re}\lambda > -1$.

1) *$L^{(\lambda)}_\ell(t)$ is the unique polynomial of degree ℓ satisfying the Laguerre differential equation*

$$\Big(t\frac{d^2}{dt^2} + (\lambda+1-t)\frac{d}{dt} + \ell\Big) f(t) = 0 \tag{3.18}$$

and

$$f^{(\ell)}(0) = (-1)^\ell. \tag{3.19}$$

2) *(recurrence relation)*

$$(\ell + t\frac{d}{dt} - t + \lambda + 1)L^{(\lambda)}_\ell(t) = (\ell+1)L^{(\lambda)}_{\ell+1}(t), \tag{3.20 a}$$

$$(\ell - t\frac{d}{dt})L^{(\lambda)}_\ell(t) = (\ell+\lambda)L^{(\lambda)}_{\ell-1}(t). \tag{3.20 b}$$

3) *(orthogonality relation)*

$$\int_0^\infty L_\ell^{(\lambda)}(t)L_s^{(\lambda)}(t)t^\lambda e^{-t}dt = \delta_{\ell s}\frac{\Gamma(\lambda+\ell+1)}{\Gamma(\ell+1)}. \tag{3.21}$$

4) *(generating function)*

$$(1-r)^{-\lambda-1}\exp\Big(\frac{rt}{r-1}\Big) = \sum_{\ell=0}^\infty L_\ell^{(\lambda)}(t)r^\ell, \quad (|r|<1). \tag{3.22}$$

5) $\{L_\ell^{(\lambda)}(t) : \ell \in \mathbb{N}\}$ *form an orthogonal basis in* $L^2(\mathbb{R}_+, t^\lambda e^{-t}dt)$ *if* λ *is real and* $\lambda > -1$.

Finally, we give a new representation of the Laguerre polynomial.

Theorem 3.11. *For any* $c \neq 0$ *and* $\ell \in \mathbb{N}$,

$$\exp\Big(-c(t\frac{d^2}{dt^2} + (\lambda+1)\frac{d}{dt})\Big)t^\ell = (-c)^\ell \ell! L_\ell^{(\lambda)}\Big(\frac{t}{c}\Big). \tag{3.23}$$

Since the differential operator

$$B_t := t\frac{d^2}{dt^2} + (\lambda+1)\frac{d}{dt}$$

is homogeneous of degree -1, namely, $B_t = cB_x$ if $x = ct$, it is sufficient to prove Theorem 3.11 in the case $c = 1$. We shall give two different proofs for this.

Proof 1. We set

$$A := t\frac{d}{dt} - \ell, \quad B := t\frac{d^2}{dt^2} + (\lambda+1)\frac{d}{dt}.$$

It follows from $[A, B] = -B$ that

$$AB^n = B^n A - nB^n$$

for all $n \in \mathbb{N}$ by induction. Then, by the Taylor expansion $e^{-B} = \sum_{n=0}^\infty \frac{(-1)^n}{n!}B^n$, we get

$$Ae^{-B} = e^{-B}A + Be^{-B}.$$

Since $At^\ell = 0$, we get $(B-A)(e^{-B}t^\ell) = 0$, namely, $e^{-B}t^\ell$ solves the Laguerre differential equation (3.18). On the other hand, $e^{-B}t^\ell$ is clearly a polynomial of t with top term t^ℓ. In view of (3.19), we have $e^{-B}t^\ell = (-1)^\ell \ell! L_\ell^{(\lambda)}(t)$. □

Proof 2. A direct computation shows

$$Bt^\ell = \ell(\lambda+\ell)t^{\ell-1}.$$

Therefore, $B^j t^\ell = 0$ for $j > \ell$ and

$$\begin{aligned} e^{-B}t^\ell &= \sum_{j=0}^\ell \frac{(-1)^j \ell(\ell-1)\cdots(\ell-j+1)(\lambda+\ell)(\lambda+\ell-1)\cdots(\lambda+\ell-j+1)}{j!}\, t^{\ell-j} \\ &= \sum_{k=0}^\ell \frac{(-1)^{\ell+k}\ell!\Gamma(\lambda+\ell+1)}{(\ell-k)!\Gamma(\lambda+k+1)}\,\frac{t^k}{k!} \\ &= (-1)^\ell \ell! L_\ell^{(\lambda)}(t). \end{aligned}$$

Hence, Theorem 3.11 has been proved. □

3.4. **Construction of an orthonormal basis in** $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$**.**
We recall from (1.2) and (2.2) that the weight function $\vartheta_{k,a}$ on $\mathbb{R}^N$ satisfies

$$\vartheta_{k,a}(x) = \|x\|^{a-2} \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, x\rangle|^{2k_\alpha} = \|x\|^{a-2}\vartheta_k(x).$$

Therefore, in the polar coordinates $x = r\omega$ ($r > 0$, $\omega \in S^{N-1}$), we have

$$\vartheta_{k,a}(x)dx = r^{2\langle k\rangle+N+a-3}\vartheta_k(\omega)drd\sigma(\omega), \tag{3.24}$$

where $d\sigma(\omega)$ is the standard measure on the unit sphere. Accordingly, we have a unitary isomorphism:

$$L^2(S^{N-1}, \vartheta_k(\omega)d\sigma(\omega)) \,\widehat{\otimes}\, L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr) \xrightarrow{\sim} L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx), \tag{3.25}$$

where $\widehat{\otimes}$ stands for the Hilbert completion of the tensor product space of two Hilbert spaces.

Combining (3.25) with Fact 2.7, we get a direct sum decomposition of the Hilbert space:

$$\sum_{m\in\mathbb{N}}^{\oplus} (\mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}) \otimes L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr) \xrightarrow{\sim} L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx). \tag{3.26}$$

In this subsection, we demonstrate the irreducible decomposition theorem of the $\mathfrak{sl}_2$ representation on (a dense subspace of) $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ by using (3.26) and finding an orthogonal basis for $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$.

For $\ell, m \in \mathbb{N}$ and $p \in \mathscr{H}_k^m(\mathbb{R}^N)$, we introduce the following functions on $\mathbb{R}^N$:

$$\Phi_\ell^{(a)}(p, \cdot) := S_a(p \otimes L_\ell^{(\lambda_{k,a,m})}). \tag{3.27}$$

Here, $S_a : C^\infty(\mathbb{R}^N) \otimes C^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}^N \setminus \{0\})$ is a linear operator defined in (3.16), $\lambda_{k,a,m} = \frac{1}{a}(2m+2\langle k\rangle+N-2)$ (see (3.11)), and $L_\ell^{(\lambda)}(t)$ is the Laguerre polynomial. Hence, for $x = r\omega \in \mathbb{R}^N$ ($r > 0$, $\omega \in S^{N-1}$), we have

$$\begin{aligned}\Phi_\ell^{(a)}(p, x) &= p(x)L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a}\|x\|^a\Big)\exp\Big(-\frac{1}{a}\|x\|^a\Big) \\ &= p(\omega)r^m L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a}r^a\Big)\exp\Big(-\frac{1}{a}r^a\Big).\end{aligned} \tag{3.28}$$

We define the following vector space of functions on $\mathbb{R}^N$ by

$$W_{k,a}(\mathbb{R}^N) := \mathbb{C}\text{-span}\{\Phi_\ell^{(a)}(p, \cdot) \mid \ell \in \mathbb{N}, m \in \mathbb{N}, p \in \mathscr{H}_k^m(\mathbb{R}^N)\}. \tag{3.29}$$

Proposition 3.12. *Suppose k is a non-negative multiplicity function on the root system $\mathscr{R}$ and $a > 0$ such that*

$$a + 2\langle k\rangle + N - 2 > 0. \tag{3.30}$$

Let $\ell, s, m, n \in \mathbb{N}$, $p \in \mathscr{H}_k^m(\mathbb{R}^N)$ and $q \in \mathscr{H}_k^n(\mathbb{R}^N)$.

1) $\Phi_\ell^{(a)}(p, x) \in C(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

2)

$$\int_{\mathbb{R}^N} \Phi_\ell^{(a)}(p, x)\overline{\Phi_s^{(a)}(q, x)}\vartheta_{k,a}(x)dx = \delta_{m,n}\delta_{\ell,s}\frac{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m} + \ell + 1)}{2^{1+\lambda_{k,a,m}}\Gamma(\ell + 1)} \int_{S^{N-1}} p(\omega)\overline{q(\omega)}\vartheta_k(\omega)d\sigma(\omega).$$

3) $W_{k,a}(\mathbb{R}^N)$ is a dense subspace of $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

Remark 3.13. *The special values of our functions $\Phi_\ell^{(a)}(p,x)$ have been used in various settings including:*

$$\begin{aligned} &a = 2 && \textit{see } [13, \S 3], \\ &k \equiv 0,\ N = 1 && \textit{see } [40], \\ &k \equiv 0,\ a = 1 && \textit{see } [36, \S 3.2]. \end{aligned}$$

Remark 3.14. *The condition* (3.30) *is automatically satisfied for $a > 0$ and a non-negative multiplicity k if $N \geq 2$.*

Proof. Our assumption (3.30) implies

$$\lambda_{k,a,m} > -1 \quad \text{for any } m \in \mathbb{N},$$

and thus $\Phi_\ell^{(a)}(p,x)$ is continuous at $x = 0$. Therefore, it is a continuous function on $x \in \mathbb{R}^N$ of exponential decay. On the other hand, we see from (3.24) that the measure $\vartheta_{k,a}(x)dx$ is locally integrable under our assumptions on a and k. Therefore, $\Phi_\ell^{(a)}(p,x) \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. Hence the first statement is proved.

To see the second and third statements, we rewrite the left-hand side of the integral as

$$\Big(\int_0^\infty L_\ell^{(\lambda_{k,a,m})}(\frac{2}{a}r^a)L_s^{(\lambda_{k,a,n})}(\frac{2}{a}r^a)\exp\Big(-\frac{2}{a}r^a\Big)r^{m+n+2\langle k\rangle+N+a-3}dr\Big)\Big(\int_{S^{N-1}} p(\omega)\overline{q(\omega)}\vartheta_k(\omega)d\sigma(\omega)\Big)$$

in the polar coordinates $x = r\omega$. Since k-harmonic polynomials of different degrees are orthogonal to each other (see Fact 2.7), the integration over S^{N-1} vanishes if $m \neq n$.

Suppose that $m = n$. By changing the variable $t := \frac{2}{a}r^a$, we see that the first integration amounts to

$$\frac{a^{\lambda_{k,a,m}}}{2^{1+\lambda_{k,a,m}}}\int_0^\infty L_\ell^{(\lambda_{k,a,m})}(t)L_s^{(\lambda_{k,a,m})}(t)\,t^{\lambda_{k,a,m}}e^{-t}dt. \tag{3.31}$$

By the orthogonality relation (3.21), we get

$$(3.31) = \delta_{\ell s}\frac{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m}+\ell+1)}{2^{1+\lambda_{k,a,m}}\Gamma(\ell+1)}.$$

Hence, the second statement is proved. The third statement follows from the completeness of the Laguerre polynomials (see Fact 3.10 4)). ◻

We pin down the following proposition which is already implicit in the proof of Proposition 3.12:

Proposition 3.15. *We fix $m \in \mathbb{N}$, $a > 0$, and a multiplicity function k satisfying*

$$2m + 2\langle k\rangle + N + a - 2 > 0.$$

We set

$$f_{\ell,m}^{(a)}(r) := \Big(\frac{2^{\lambda_{k,a,m}+1}\Gamma(\ell+1)}{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m}+\ell+1)}\Big)^{1/2} r^m L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a}r^a\Big)\exp(-\frac{1}{a}r^a) \quad \textit{for } \ell \in \mathbb{N}. \tag{3.32}$$

Then $\{f_{\ell,m}^{(a)}(r) : \ell \in \mathbb{N}\}$ forms an orthonormal basis in $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$.

Remark 3.16. *Let $c_0, c_1, \dots$ be a sequence of positive real numbers. Fix a parameter $\alpha > 0$. Dunkl* [12] *proves that the only possible orthogonal sets $\{L_\ell^{(\alpha)}(c_\ell r)\exp(-\frac{1}{2}c_\ell r)\}_{\ell=0}^{\infty}$ for the measure $r^{\alpha+\mu}dr$ on $\mathbb{R}_+$, with $\mu \geq 0$, are the two cases (1) $\mu = 0$, $c_\ell = c_0$ for all ℓ; (2) $\mu = 1$, $c_\ell = c_0 \frac{\alpha+1}{\alpha+2\ell+1}$.*

For each $m \in \mathbb{N}$, we take an orthonormal basis $\{h_j^{(m)}\}_{j \in J_m}$ of the space $\mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}$. Proposition 3.12 immediately yields the following statement.

Corollary 3.17. *Suppose that $a > 0$ and that the non-negative multiplicity function k satisfies the inequality* (3.30)*. For $\ell, m \in \mathbb{N}$ and $j \in J_m$, we set*

$$\Phi_{\ell,m,j}^{(a)}(x) := h_j^{(m)}\Big(\frac{x}{\|x\|}\Big) f_{\ell,m}^{(a)}(\|x\|).$$

Then, the set $\left\{\Phi_{\ell,m,j}^{(a)} \mid \ell \in \mathbb{N}, m \in \mathbb{N}, j \in J_m\right\}$ forms an orthonormal basis of $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

Remark 3.18. *A basis of $\mathscr{H}_k^m(\mathbb{R}^N)$ is constructed in* [14, Corollary 5.1.13].

3.5. $\mathfrak{sl}_2$ **representation on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.**
Now we are ready to exhibit the action of the $\mathfrak{sl}_2$ triple $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$ on the basis $\Phi_\ell^{(a)}(p, \cdot)$ (see (3.8 a–c) and (3.28) for the definitions). We recall from (3.9 a–c) that $\widetilde{\mathbb{H}}_{k,a} = \omega_{k,a}(\mathbf{k})$, $\widetilde{\mathbb{E}}_{k,a}^+ = \omega_{k,a}(\mathbf{n}^+)$, and $\widetilde{\mathbb{E}}_{k,a}^- = \omega_{k,a}(\mathbf{n}^-)$.

Theorem 3.19. *Let $W_{k,a}(\mathbb{R}^N)$ be the dense subspace of $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ defined in* (3.29)*. Then, $W_{k,a}(\mathbb{R}^N)$ is stable under the action of $\mathfrak{sl}(2, \mathbb{C})$. More precisely, for each fixed $p \in \mathscr{H}_k^m(\mathbb{R}^N)$, the action $\omega_{k,a}$ (see* (3.9 a–c)*) is given as follows:*

$$\omega_{k,a}(\mathbf{k})\Phi_\ell^{(a)}(p, x) = (2\ell + \lambda_{k,a,m} + 1)\Phi_\ell^{(a)}(p, x), \tag{3.33 a}$$

$$\omega_{k,a}(\mathbf{n}^+)\Phi_\ell^{(a)}(p, x) = i(\ell + 1)\Phi_{\ell+1}^{(a)}(p, x), \tag{3.33 b}$$

$$\omega_{k,a}(\mathbf{n}^-)\Phi_\ell^{(a)}(p, x) = i(\ell + \lambda_{k,a,m})\Phi_{\ell-1}^{(a)}(p, x), \tag{3.33 c}$$

where $\Phi_\ell^{(a)}(p, x)$ is defined in (3.28) *and $\lambda_{k,a,m} = (2m + 2\langle k\rangle + N - 2)/a$ (see* (3.11)*). We have used the convention $\Phi_{-1}^{(a)} \equiv 0$.*

Theorem 3.19 may be visualized by the diagram below. We see that for each fixed k, a, and $p \in \mathscr{H}_k^m(\mathbb{R}^N)$, the operators $\omega_{k,a}(\mathbf{n}^+)$ and $\omega_{k,a}(\mathbf{n}^-)$ act as *raising/lowering operators*.

$$\begin{tikzcd}
\cdots & \circ \arrow[r, bend left] & \overset{\ell-1}{\circ} \arrow[l, bend left] \arrow[r, bend left, "\mathbf{n}^+"] & \overset{\ell}{\circ} \arrow[l, bend left, "\mathbf{n}^-"] \arrow[r, bend left, "\mathbf{n}^+"] & \overset{\ell+1}{\circ} \arrow[l, bend left, "\mathbf{n}^-"] \arrow[r, bend left] & \circ \arrow[l, bend left] & \cdots
\end{tikzcd}$$

Diagram 3.5.

Here, the dots represent $\omega_{k,a}(\mathbf{k})$ eigenvectors $\Phi_\ell^{(a)}(p, x)$ arranged by increasing $\omega_{k,a}(\mathbf{k})$ eigenvalues, from left to right.

Proof of Theorem 3.19. For simplicity, we use the notation $P_{t,\lambda} = t\frac{d^2}{dt^2} + (\lambda_{k,a,m} + 1 - t)\frac{d}{dt}$ as in (3.17), where λ stands for $\lambda_{k,a,m}$. By the formula $\Phi_\ell^{(a)}(p, \cdot) = S_a(p \otimes L_\ell^{(\lambda)})$ (see (3.27)) and by Lemma 3.9, it is sufficient to prove

$$(-2P_{t,\lambda} + (\lambda + 1))L_\ell^{(\lambda)} = (2\ell + \lambda + 1)L_\ell^{(\lambda)}, \tag{3.34 a}$$

$$(-i(P_{t,\lambda} - t\frac{d}{dt} + t - \lambda - 1))L_\ell^{(\lambda)} = i(\ell + 1)L_{\ell+1}^{(\lambda)}, \tag{3.34 b}$$

$$-i(P_{t,\lambda} + t\frac{d}{dt})L_\ell^{(\lambda)} = i(\ell + \lambda)L_{\ell-1}^{(\lambda)}. \tag{3.34 c}$$

Since the Laguerre polynomial $L_\ell^{(\lambda)}(t)$ satisfies the Laguerre differential equation

$$P_{t,\lambda}L_\ell^{(\lambda)}(t) = -\ell L_\ell^{(\lambda)}(t)$$

(see (3.18)), the assertion (3.34 a) is now clear. The assertions (3.34 b) and (3.34 c) are reduced to the recurrence relations (3.20 a) and (3.20 b), respectively. □

Remark 3.20. *An alternative proof of* (3.33 a) *will be given in Section 5.4 (see Remark 5.17).*

By using the orthonormal basis $\{f_{\ell,m}^{(a)}(r)\}$ (see (3.32)), we normalize $\Phi_\ell^{(a)}(p, x)$ as

$$\begin{aligned} \widetilde{\Phi}_\ell^{(a)}(p, x) &:= f_{\ell,m}^{(a)}(r)p(\omega) \\ &= \Big(\frac{2^{\lambda_{k,a,m}+1}\Gamma(\ell + 1)}{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m} + \ell + 1)}\Big)^{\frac{1}{2}}\Phi_\ell^{(a)}(p, x) \end{aligned} \tag{3.35}$$

for $x = r\omega$ ($r > 0$, $\omega \in S^{N-1}$). Then, Theorem 3.19 is reformulated as follows:

Theorem 3.21. *For any $p \in \mathscr{H}_k^m(\mathbb{R}^N)$, we have*

$$\omega_{k,a}(\mathbf{k})\widetilde{\Phi}_\ell^{(a)}(p, x) = (2\ell + \lambda_{k,a,m} + 1)\widetilde{\Phi}_\ell^{(a)}(p, x), \tag{3.36 a}$$

$$\omega_{k,a}(\mathbf{n}^+)\widetilde{\Phi}_\ell^{(a)}(p, x) = i\sqrt{(\ell + 1)(\lambda_{k,a,m} + \ell + 1)}\,\widetilde{\Phi}_{\ell+1}^{(a)}(p, x), \tag{3.36 b}$$

$$\omega_{k,a}(\mathbf{n}^-)\widetilde{\Phi}_\ell^{(a)}(p, x) = i\sqrt{(\lambda_{k,a,m} + \ell)\ell}\,\widetilde{\Phi}_{\ell-1}^{(a)}(p, x). \tag{3.36 c}$$

We recall that an operator T densely defined on a Hilbert space is called essentially self-adjoint, if it is symmetric and its closure is a self-adjoint operator.

Corollary 3.22. *Let $a > 0$ and k be a non-negative multiplicity function satisfying* (3.30).

1) *The differential-difference operator $\Delta_{k,a} = \|x\|^{2-a}\Delta_k - \|x\|^a$ is an essentially self-adjoint operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*
2) *There is no continuous spectrum of $\Delta_{k,a}$.*
3) *The set of discrete spectra of $-\Delta_{k,a}$ is given by*

$$\begin{aligned} &\{2a\ell + 2m + 2\langle k\rangle + N - 2 + a : \ell, m \in \mathbb{N}\} \quad (N \geq 2), \\ &\{2a\ell + 2\langle k\rangle + a \pm 1 : \ell \in \mathbb{N}\} \quad (N = 1). \end{aligned}$$

Proof. In light of the formula (3.9 a)

$$\Delta_{k,a} = -a\omega_{k,a}(\mathbf{k}),$$

the eigenvalues of $\Delta_{k,a}$ are read from Theorem 3.19. Since $W_{k,a}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ (see Proposition 3.12), the remaining statement of Corollary 3.22 is straightforward from the following fact. □

Fact 3.23. *Let T be a symmetric operator on a Hilbert space $\mathscr{H}$ with domain $\mathbb{D}(T)$, and let $\{f_n\}_n$ be a complete orthogonal set in $\mathscr{H}$. If each $f_n \in \mathbb{D}(T)$ and there exists $\mu_n \in \mathbb{R}$ such that $T f_n = \mu_n f_n$, for every n, then T is essentially self-adjoint.*

Remark 3.24. *We shall see in Theorem 3.30 that the action of $\mathfrak{sl}(2,\mathbb{R})$ in Theorem 3.21 lifts to a unitary representation of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ and that Corollary 3.22 1) is a special case of the general theory of discretely decomposable $(\mathfrak{g}_{\mathbb{C}}, K)$-modules (see* [33, 34]*).*

3.6. **Discretely decomposable representations.**

Theorem 3.19 asserts that $W_{k,a}(\mathbb{R}^N)$ is an $\mathfrak{sl}(2,\mathbb{C})$-invariant, dense subspace in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. For $N > 1$, this is a 'huge' representation in the sense that it contains an infinitely many inequivalent irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$.

By a theorem of Harish-Chandra, Lepowsky and Rader, any irreducible, infinitesimally unitary $(\mathfrak{g}_{\mathbb{C}}, K)$-module is the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of a (unique) irreducible unitary representation of G (see [32, Theorem 0.6]). This result was generalized to a discretely decomposable $(\mathfrak{g}_{\mathbb{C}}, K)$-modules by the second-named author (see [34, Theorem 2.7]).

In this section, we discuss the meaning of Theorem 3.19 from the point of view of discretely decomposable representations.

We begin with a general setting. Let G be a semisimple Lie group, and K a maximal compact subgroup of G (modulo the center of G). We write $\mathfrak{g}$ for the Lie algebra of G, and $\mathfrak{g}_{\mathbb{C}}$ for its complexification. The following notion singles out an algebraic property of unitary representations that split into irreducible representations without continuous spectra.

Definition 3.25. *Let (ϖ, X) be a $(\mathfrak{g}_{\mathbb{C}}, K)$-module.*

(1) ([33, Part I, §1]) *We say ϖ is K-admissible if* $\dim \mathrm{Hom}_K(\tau, \varpi) < \infty$ *for any $\tau \in \widehat{K}$.*

(2) ([33, Part III, Definition 1.1]) *We say ϖ is a discretely decomposable if there exist a sequence of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules X_j such that*

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \cdots, \quad X = \bigcup_{j=0}^{\infty} X_j,$$

$$X_j/X_{j-1} \text{ is of finite length as a } (\mathfrak{g}_{\mathbb{C}}, K)\text{-module for } j = 1, 2, \ldots.$$

(3) *We say ϖ is infinitesimally unitarizable if there exists a Hermitian inner product $(\ ,\ )$ on X such that*

$$(\varpi(Y)u, v) = -(u, \varpi(Y)v) \quad \text{for any } Y \in \mathfrak{g}, \text{ and any } u, v \in X.$$

We collect some basic results on discretely decomposable $(\mathfrak{g}_{\mathbb{C}}, K)$-modules:

Fact 3.26 (see [33, 34])**.** *Let (ϖ, X) be a $(\mathfrak{g}_C, K)$-module.*

1) *If ϖ is K-admissible, then ϖ is discretely decomposable as a $(\mathfrak{g}_{\mathbb{C}}, K)$-modules.*

2) *Suppose ϖ is discretely decomposable as a $(\mathfrak{g}_{\mathbb{C}}, K)$-module. If ϖ is infinitesimally unitarizable, then ϖ is isomorphic to an algebraic direct sum of irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules.*

3) *Any discretely decomposable, infinitesimally unitary* $(\mathfrak{g}_{\mathbb{C}}, K)$*-module is the underlying* $(\mathfrak{g}_{\mathbb{C}}, K)$*-module of a unitary representation of* G*. Furthermore, such a unitary representation is unique.*

We shall apply this concept to the specific situation where $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and G is the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$.

We recall from (3.8 a) that

$$\mathbf{k} = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\,(\mathbf{e}^- - \mathbf{e}^+) \in \mathfrak{sl}(2,\mathbb{C}).$$

Let $\mathfrak{k} := \mathbb{R}(\mathbf{e}^- - \mathbf{e}^+) = i\,\mathbb{R}\mathbf{k}$ and K be the subgroup of G with Lie algebra $\mathfrak{k}$. Since G is taken to be simply connected, the exponential map

$$\mathbb{R} \to K, \quad t \mapsto \operatorname{Exp}(it\mathbf{k})$$

is a diffeomorphism.

For $z \in i\,\mathbb{R}$, we set

$$\gamma_z := \operatorname{Exp}(-z\mathbf{k}) = \operatorname{Exp}\begin{pmatrix} 0 & iz \\ -iz & 0 \end{pmatrix} \in K. \tag{3.37}$$

Since $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$ forms an $\mathfrak{sl}_2$ triple, we have

$$\operatorname{Ad}(\gamma_z)\mathbf{n}^+ = e^{-2z}\mathbf{n}^+, \quad \operatorname{Ad}(\gamma_z)\mathbf{n}^- = e^{2z}\mathbf{n}^-.$$

Then it is easy to see that the subgroup

$$C(G) := \{\gamma_{n\pi i} : n \in \mathbb{Z}\} \simeq \mathbb{Z} \tag{3.38}$$

coincides with the center of G.

Next, we give a parametrization of one-dimensional representations of $K \simeq \mathbb{R}$ as

$$\widehat{K} \simeq \mathbb{C}, \quad \chi_\mu \leftrightarrow \mu \tag{3.39}$$

by the formula $\chi_\mu(\gamma_z) = e^{-\mu z}$ or equivalently, $d\chi_\mu(\mathbf{k}) = \mu$.

We shall call χ_μ simply as the K-type μ.

Let (ϖ, X) be a $(\mathfrak{g}_{\mathbb{C}}, K)$-module. A non-zero vector $v \in X$ is a *lowest weight vector* of weight $\mu \in \mathbb{C}$ if v satisfies

$$\varpi(\mathbf{n}^-)v = 0, \quad \text{and} \quad \varpi(\mathbf{k})v = \mu v.$$

We say (ϖ, V) is a *lowest weight module* of weight μ if V is generated by such v. For each $\lambda \in \mathbb{C}$, there exists a unique irreducible lowest weight $(\mathfrak{g}_{\mathbb{C}}, K)$-module, to be denoted by $\pi_K(\lambda)$, of weight $\lambda + 1$.

With this normalization, we pin down the following well-known properties of the $(\mathfrak{g}_{\mathbb{C}}, K)$-module $\pi_K(\lambda)$ for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$:

Fact 3.27.

1) *For a real* λ *with* $\lambda \geq -1$*, there exists a unique unitary representation, denoted by* $\pi(\lambda)$*, of* $G = \widetilde{SL(2,\mathbb{R})}$ *such that its underlying* $(\mathfrak{g}_{\mathbb{C}}, K)$*-module is isomorphic to* $\pi_K(\lambda)$*.*
2) $\pi(-1)$ *is the trivial one-dimensional representation.*
3) *For* $\lambda > 0$*,* $\pi(\lambda)$ *is a relative discrete series representation, namely, its matrix coefficients are square integrable over* G *modulo its center* $C(G)$*.*

4) $\pi(\frac{1}{2}) \oplus \pi(-\frac{1}{2})$ *is the Weil representation of* $Mp(1,\mathbb{R})$*, the two fold covering group of* $SL(2,\mathbb{R})$*.*
5) $\gamma_{\pi i}$ *(see* (3.37)*) acts on* $\pi(\lambda)$ *as scalar* $e^{-\pi i(\lambda+1)}$*.*
6) $\pi(\lambda)$ *is well-defined as a unitary representation of* $Mp(1,\mathbb{R})$ *if* $\lambda \in \frac{1}{2}\mathbb{Z}$*, of* $SL(2,\mathbb{R})$ *if* $\lambda \in \mathbb{Z}$*, and of* $PSL(2,\mathbb{R})$ *if* $\lambda \in 2\mathbb{Z}+1$*.*
7) *For* $\lambda \neq -1,-3,-5,\ldots$, $\pi_K(\lambda)$ *is an infinite dimensional representation. For* $\lambda > -1$*, we fix a* G*-invariant inner product on the representation space of* $\pi(\lambda)$*. Then we can find an orthonormal basis* $\{v_\ell : \ell \in \mathbb{N}\}$ *such that*
$$\begin{aligned}\pi_K(\lambda)(\mathbf{k})v_\ell &= (2\ell+\lambda+1)v_\ell,\\ \pi_K(\lambda)(\mathbf{n}^+)v_\ell &= i\sqrt{(\ell+1)(\lambda+\ell+1)}\,v_{\ell+1},\\ \pi_K(\lambda)(\mathbf{n}^-)v_\ell &= i\sqrt{(\lambda+\ell)\ell}\,v_{\ell-1}.\end{aligned}$$
Here, we set $v_{-1} = \{0\}$*. In particular,* $\pi_K(\lambda)$ *has the* K*-type* $\{\lambda+1, \lambda+3, \lambda+5, \ldots\}$ *with respect to the parametrization* (3.39)*.*
8) *For* $\lambda = -m$ $(m = 1,2,\ldots)$*,* $\pi_K(\lambda)$ *is an* m*-dimensional irreducible representation of* $\mathfrak{sl}(2,\mathbb{C})$*.*

By using Fact 3.27, we can read from the formulas in Theorem 3.21 the following statement:

Theorem 3.28. *Suppose* a *is a non-zero complex number and* k *is a non-negative root multiplicity function satisfying the inequality* (3.30)*, i.e.* $a + 2\langle k\rangle + N - 2 > 0$*.*
1) $(\omega_{k,a}, W_{k,a}(\mathbb{R}^N))$ *is a* $\mathfrak{C} \times (\mathfrak{g}_\mathbb{C}, K)$*-module.*
2) *As a* $(\mathfrak{g}_\mathbb{C}, K)$*-module,* $\omega_{k,a}$ *is* K*-admissible and hence discretely decomposable (see Definition 3.25).*
3) $(\omega_{k,a}, W_{k,a}(\mathbb{R}^N))$ *is decomposed into the direct sum of* $\mathfrak{C} \times (\mathfrak{g}_\mathbb{C}, K)$*-modules as follows:*
$$W_{k,a}(\mathbb{R}^N) \simeq \bigoplus_{m=0}^{\infty} \mathscr{H}_k^m(\mathbb{R}^N)\big|_{S^{N-1}} \otimes \pi_K(\lambda_{k,a,m}). \tag{3.40}$$
Here, $\lambda_{k,a,m} = \frac{2m+2\langle k\rangle+N-2}{a}$ *(see* (3.11)*). The Coxeter group* $\mathfrak{C}$ *acts on the first factor, and the Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$ *acts on the second factor of each summand in* (3.40)*.*

Proof of Theorem 3.28. We fix a non-zero $p \in \mathscr{H}_k^m(\mathbb{R}^N)$. Then, it follows from Theorem 3.19 and Fact 3.27 that for $\mathfrak{sl}(2,\mathbb{R})$ acts on the vector space
$$\mathbb{C}\text{-span}\{\Phi_\ell^{(a)}(p,\cdot) : \ell \in \mathbb{N}\}$$
as an irreducible lowest weight module $\pi_K(\lambda_{k,a,m})$. By (3.26), we get the isomorphism (3.40) as $(\mathfrak{g}_\mathbb{C}, K)$-modules.

On the other hand, the Coxeter group $\mathfrak{C}$ leaves $\mathscr{H}_k^m(\mathbb{R}^N)$ invariant. Furthermore, as we saw in Lemma 3.4, the action of $\mathfrak{C}$ and $\mathfrak{sl}(2,\mathbb{R})$ commute with each other. Hence, the first and third statements are proved.

It follows from the decomposition formula (3.40) that $\omega_{k,a}$ is K-admissible because the K-type of an individual $\pi_K(\lambda)$ is of the form $\{\lambda+1, \lambda+3, \ldots\}$ by Fact 3.27, $\lambda_{k,a,m}$ increases as m increases, and $\dim \mathscr{H}_k^m(\mathbb{R}^N) < \infty$. Hence, the second statement is also proved. $\square$

For $f, g \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$, we write its inner product as

$$\langle\!\langle f, g \rangle\!\rangle_k := \int_{\mathbb{R}^N} f(x)\overline{g(x)}\,\vartheta_{k,a}(x)dx. \tag{3.41}$$

Proposition 3.29. *Suppose that $a > 0$ and that k is a non-negative multiplicity function such that $a+2\langle k\rangle+N-2 > 0$. Then, the representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$ on $W_{k,a}(\mathbb{R}^N)$ is infinitesimally unitary with respect to the inner product $\langle\!\langle\ ,\ \rangle\!\rangle_k$, namely,*

$$\langle\!\langle \omega_{k,a}(X) f, g \rangle\!\rangle_k = -\langle\!\langle f, \omega_{k,a}(X) g \rangle\!\rangle_k$$

for any $X \in \mathfrak{sl}(2,\mathbb{R})$ and $f, g \in W_{k,a}(\mathbb{R}^N)$.

Proof. As we saw in (2.4) that the Dunkl operators are skew-symmetric with respect to the measure $\vartheta_k(x)dx$. In view of the definitions of Δ_k (see (2.9)) and $\mathbb{E}^-_{k,a} = \frac{i}{a}\|x\|^{2-a}\Delta_k$ (see (3.3)), we see that $\mathbb{E}^-_{k,a}$ is a skew-symmetric operator with respect to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_k$. Likewise for $\mathbb{E}^+_{k,a}$. Further, the commutation relation $\mathbb{H}_{k,a} = [\mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}]$ shows that $\mathbb{H}_{k,a}$ is also skew-symmetric. Thus, for all $X \in \mathfrak{sl}(2,\mathbb{R})$, $\omega_{k,a}(X)$ is skew-symmetric. □

3.7. The integrability of the representation $\omega_{k,a}$.

Applying the general result on discretely decomposable representations (see Fact 3.26) to our specific setting where G is the universal covering group of $SL(2,\mathbb{R})$, we get the following two theorems:

Theorem 3.30. *Suppose $a > 0$ and k is a non-negative multiplicity function satisfying*

$$a + 2\langle k\rangle + N - 2 > 0. \tag{3.42}$$

Then the infinitesimal representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$ lifts to a unique unitary representation, to be denoted by $\Omega_{k,a}$, of G on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. In particular, we have

$$\omega_{k,a}(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Omega_{k,a}(\mathrm{Exp}(tX)), \qquad X \in \mathfrak{g},$$

on $W_{k,a}(\mathbb{R}^N)$, the dense subspace (3.29) *of $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. Here, we have written* Exp *for the exponential map of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ into G.*

Theorem 3.31. *Retain the assumption of Theorem 3.30. Then, as a representation of the direct product group $\mathfrak{C} \times G$, the unitary representation $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ decomposes discretely as*

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) = \sum_{m=0}^{\infty}{}^{\oplus} \Big(\mathscr{H}^m_k(\mathbb{R}^N)|_{S^{N-1}}\Big) \otimes \pi(\lambda_{k,a,m}). \tag{3.43}$$

Here, we recall that $\mathfrak{C}$ is the Coxeter group of the root system, G is the universal covering group of $SL(2,\mathbb{R})$, and $\lambda_{k,a,m} = \frac{2m+2\langle k\rangle+N-2}{a}$ (see (3.11)*). The decomposition* (3.43) *of the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is given by the formula* (3.26) *and $\pi(\lambda)$ $(\lambda > -1)$ is the irreducible unitary representation of $\widetilde{SL(2,\mathbb{R})}$ described in Fact 3.27. In particular, the summands are mutually orthogonal with respect to the inner product* (3.41) *on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*

Remark 3.32 (The $N = 1$ case)**.** *In* [40] *Kostant exhibits a family of representations with continuous parameter of $\mathfrak{sl}(2,\mathbb{R})$ by second order differential operators on* $(0,\infty)$*. He uses Nelson's result* [44] *to study the exponentiation of such representations. See also* [47].

In the $N = 1$ case, the decomposition in Theorem 3.28 (and hence in Theorem 3.31) is reduced to a finite sum because $\mathscr{H}_k^m(\mathbb{R}^N) = 0$ *if* $m \geq 2$ *and* $N = 1$. *Indeed, there are two summands according to even ($m = 0$) and odd ($m = 1$) functions. We observe that the difference operator in* (2.10) *vanishes on even functions of one variable, so the Dunkl Laplacian* Δ_k *collapses to the differential operator* $\frac{d^2}{dx^2} + \frac{2k}{x}\frac{d}{dx}$. *Thus, our generators* (3.3) *acting on even functions on* $(0, \infty)$ *take the form*

$$\mathbb{H}_{k,a} = \frac{2}{a}x\frac{d}{dx} + \frac{2k+a-1}{a}, \qquad \mathbb{E}^+_{k,a} = \frac{i}{a}x^a, \qquad \mathbb{E}^-_{k,a} = \frac{i}{a}x^{2-a}\Big(\frac{d^2}{dx^2} + \frac{2k}{x}\frac{d}{dx}\Big).$$

We may compare these with the generators in Kostant's paper [40], *where his generators on* $(0, \infty)$ *are*

$$iy, \qquad 2y\frac{d}{dy} + 1, \qquad i\Big(y\frac{d^2}{dy^2} + \frac{d}{dy} - \frac{r^2}{4y}\Big),$$

which by the substitution $y = \frac{1}{a}x^a$ *and* $\varphi(y) = x^{k-\frac{1}{2}}\varphi(x)$ *become our operators* $\{\mathbb{H}_{k,a}, \mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}\}$ *with* $r = \frac{2k-1}{a}$. *Note that our generators acting on odd functions do not appear in Kostant's picture.*

Remark 3.33. *The assumption* $a + 2\langle k\rangle + N - 2 > 0$ *implies*

$$\lambda_{k,a,m} > -1 \quad \text{for any } m \in \mathbb{N},$$

whence there exists an irreducible, infinite dimensional unitary representation $\pi(\lambda_{k,a,m})$ *of* G *such that its underlying* $(\mathfrak{g}_{\mathbb{C}}, K)$*-module is isomorphic to* $\pi_K(\lambda_{k,a,m})$ *by Fact 3.27 (1).*

By the explicit construction of the direct summand in Theorem 3.19, we have

Corollary 3.34. *As a representation of* $\widetilde{SL(2,\mathbb{R})}$, *minimal K-types of the irreducible summands in* (3.43) *are given by*

$$h(x)\exp\Big(-\frac{1}{a}\|x\|^a\Big), \quad h \in \mathscr{H}_k^m(\mathbb{R}^N).$$

As we have seen that $\omega_{k,a}$ lifts to the unitary representation $\Omega_{k,a}$ of the universal covering group $G = \widetilde{SL(2,\mathbb{R})}$ for any k and a with certain positivity (3.42). On the other hand, if k and a satisfies a certain rational condition (see below), then $\Omega_{k,a}$ is well-defined for some finite covering groups of $PSL(2,\mathbb{R})$. This representation theoretic observation gives an explicit formula of the order of the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ (see Section 5). We pin down a precise statement here.

Proposition 3.35. *Retain the notation of Theorem 3.30, and recall the definition of the index* $\langle k\rangle$ *from* (2.3). *Then the unitary representation* $\Omega_{k,a}$ *of the universal covering group* G *of* $SL(2,\mathbb{R})$ *is well-defined also as a representation of some finite covering group of* $PSL(2,\mathbb{R})$ *if and only if both* a *and* $\langle k\rangle$ *are rational numbers.*

Proof. It follows from Fact 3.27 5) that the central element $\gamma_{n\pi i} \in C(G)$ acts on $\pi(\lambda_{k,a,m})$ by the scalar

$$e^{-\pi i n(\lambda_{k,a,m}+1)} = \exp\Big(-\frac{n}{a}\cdot 2\pi m i\Big)\exp\Big(-\frac{N+2\langle k\rangle+a-2}{a}n\pi i\Big).$$

This equals 1 for all m if and only if

$$\frac{n}{a} \in \mathbb{Z} \quad \text{and} \quad \frac{n(2\langle k\rangle + N - 2 + a)}{a} \in 2\mathbb{Z}.$$

It is easy to see that there exists a non-zero integer n satisfying these two conditions if and only if both a and $\langle k\rangle$ are rational numbers. For such n, $\Omega_{k,a}$ is well-defined for $G/n\mathbb{Z}$. Hence, Proposition 3.35 is proved. □

We recall from (3.38) that we have identified the center $C(G)$ of the simply-connected Lie group $G = \widetilde{SL(2,\mathbb{R})}$ with the integer group $\mathbb{Z}$. Then, we have

$$PSL(2,\mathbb{R}) \simeq G/\mathbb{Z}, \quad SL(2,\mathbb{R}) \simeq G/2\mathbb{Z}, \quad Mp(1,\mathbb{R}) \simeq G/4\mathbb{Z}.$$

As a special case of Proposition 3.35 and its proof, we have:

Remark 3.36. *Let $\Omega_{k,a}$ be the unitary representation of the universal covering group G.*

(1) *Suppose $a = 2$.*
  (a) *$\Omega_{k,2}$ descends to $SL(2,\mathbb{R})$ if and only if $2\langle k\rangle + N$ is an even integer.*
  (b) *$\Omega_{k,2}$ descends to $Mp(1,\mathbb{R})$ if and only if $2\langle k\rangle + N$ is an integer.*
  This compares well with the Schrödinger model on $L^2(\mathbb{R}^N)$ of the Weil representation $\Omega_{0,2}$ of the metaplectic group $Mp(N,\mathbb{R})$ and its restriction to a subgroup locally isomorphic to $SL(2,\mathbb{R})$ (cf. [59] *and* [30]*).*
(2) *Suppose $a = 1$.*
  (a) *$\Omega_{k,1}$ descends to $PSL(2,\mathbb{R})$ if and only if $2\langle k\rangle + N$ is an odd integer.*
  (b) *$\Omega_{k,1}$ descends to $SL(2,\mathbb{R})$ if and only if $2\langle k\rangle$ is an integer.*
  (c) *$\Omega_{k,1}$ descends to $Mp(1,\mathbb{R})$ if and only if $4\langle k\rangle$ is an integer.*
  The case $k \equiv 0$ corresponds to the Schrödinger model on $L^2(\mathbb{R}^N, \frac{dx}{\|x\|})$ of the minimal representation $\Omega_{0,1}$ of the conformal group and its restriction to a subgroup locally isomorphic to $SL(2,\mathbb{R})$ (cf. [39]*).*

Remark 3.37. *C. Dunkl reminded us of that the parity condition of $2\langle k\rangle + N$ appeared also in a different context, i.e., in* [15, Lemma 5.1]*, where the authors investigated a sufficient condition on k for the existence and uniqueness of expanding a homogenous polynomial in terms of k-harmonics.*

3.8. **Connection with the Gelfand–Gindikin program.**

We consider the following closed cone in $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ defined by

$$W := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc \le 0,\ b \ge c \right\}.$$

Then, W is $SL(2,\mathbb{R})$-invariant and is expressed as

$$W = i\,\mathrm{Ad}(SL(2,\mathbb{R}))\mathbb{R}_{\ge 0}\mathbf{k}.$$

We write $\exp_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to SL(2,\mathbb{C})$ for the exponential map. Its restriction to iW is an injective map, and we define the following subset $\Gamma(W)$ of $SL(2,\mathbb{C})$ by

$$\Gamma(W) := SL(2,\mathbb{R}) \exp_{\mathbb{C}}(iW).$$

Since W is $SL(2,\mathbb{R})$-invariant, $\Gamma(W)$ becomes a semigroup, sometimes referred to as the *Olshanski semigroup*.

Denote by $\widetilde{\Gamma(W)}$ the universal covering semigroup of $\Gamma(W)$, and write

$$\mathrm{Exp} : \mathfrak{g} + iW \to \widetilde{\Gamma(W)}$$

for the lifting of $\exp_{\mathbb{C}}|_{\mathfrak{g}+iW} : \mathfrak{g} + iW \to \Gamma(W)$. Then $\widetilde{\Gamma(W)} = \widetilde{SL(2,\mathbb{R})}\,\mathrm{Exp}(iW)$ and the polar map

$$\widetilde{SL(2,\mathbb{R})} \times W \to \widetilde{\Gamma(W)},\ (g, X) \mapsto g\,\mathrm{Exp}(iX)$$

is a homeomorphism.

Since W is an $\mathrm{Ad}(SL(2,\mathbb{R}))$-invariant cone, $\Gamma(W)$ is invariant under the action of $SL(2,\mathbb{R})$ from the left and right. Thus, the semigroup $\Gamma(W)$ is written also as

$$\Gamma(W) = SL(2,\mathbb{R})\exp_{\mathbb{C}}(-\mathbb{R}_{\geq 0}\mathbf{k})SL(2,\mathbb{R}).$$

Its interior is given by

$$\Gamma(W^0) = SL(2,\mathbb{R})\exp(-\mathbb{R}_{+}\mathbf{k})SL(2,\mathbb{R}).$$

See [26, Theorem 7.25]. Accordingly, we have

$$\widetilde{\Gamma(W)} = \widetilde{SL(2,\mathbb{R})}\exp_{\mathbb{C}}(-\mathbb{R}_{\geq 0}\mathbf{k})\widetilde{SL(2,\mathbb{R})}.$$

By Theorem 3.19, $\Omega_{k,a}$ is a discretely decomposable unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. It has a lowest weight $(2\langle k\rangle + N + a - 2)/a$. It then follows from [27, Theorem B] that $\Omega_{k,a}$ extends to a representation of the Olshanski semigroup $\widetilde{\Gamma(W)}$, denoted by the same symbol $\Omega_{k,a}$, such that:

(P1) $\Omega_{k,a} : \widetilde{\Gamma(W)} \to \mathscr{B}(L^2)$ is strongly continuous semigroup homomorphism.
(P2) For all $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$, the map $\gamma \mapsto \langle\langle \Omega_{k,a}(\gamma)f, f\rangle\rangle_k$ is holomorphic in the interior of $\widetilde{\Gamma(W)}$.
(P3) $\Omega_{k,a}(\gamma)^* = \Omega_{k,a}(\gamma^\sharp)$, where $\gamma^\sharp = \mathrm{Exp}(iX)g^{-1}$ for $\gamma = g\,\mathrm{Exp}(iX)$.

Here, we have denoted by $\mathscr{B}(L^2)$ the space of bounded operators on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

Remark 3.38. *The Gelfand–Gindikin program* [21] *seeks for the understanding of a 'family of irreducible representations' by using complex geometric methods. This program has been particularly developed for lowest weight representations by Olshanski* [46] *and Stanton* [54]*, Hilgert, Neeb* [26]*, and some others. The study of our holomorphic semigroup $\Omega_{k,a}$ by using the Olshanski semigroup $\widetilde{\Gamma(W)}$ may be regarded as a descendant of this program.*

Henceforth we will use the notation $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$ and $\mathbb{C}^{++} := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$.

For $z \in \mathbb{C}^+$, we extend the one-parameter subgroup γ_z ($z \in i\mathbb{R}$) (see (3.37)) holomorphically as

$$\gamma_z := \mathrm{Exp}(-z\mathbf{k}) = \mathrm{Exp}(iz\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \in \widetilde{\Gamma(W)}. \tag{3.44}$$

Then, the operators $\Omega_{k,a}(\gamma_z)$ have the following property:

$$\Omega_{k,a}(\gamma_{z_1})\,\Omega_{k,a}(\gamma_{z_2}) = \Omega_{k,a}(\gamma_{z_1+z_2}), \qquad \forall z_1, z_2 \in \mathbb{C}^+,$$
$$\Omega_{k,a}(\gamma_z)^* = \Omega(\gamma_{\bar{z}}), \qquad z \in \mathbb{C}^+,$$
$$\Omega_{k,a}(\gamma_0) = \mathrm{id}\,.$$

Following the formulation of [36, Proposition 3.6.1] (the case $k \equiv 0$, $a = 1$), we summarize basic properties of the holomorphic representation $\Omega_{k,a}$ of the semigroup $\widetilde{\Gamma(W)}$.

Theorem 3.39. *Suppose $a > 0$ and k is a non-negative multiplicity function on the root system satisfying* (3.30)*, i.e.* $a + 2\langle k\rangle + N - 2 > 0$*.*

(1) *The map*

$$\widetilde{\Gamma(W)} \times L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \longrightarrow L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx), \quad (\gamma, f) \mapsto \Omega_{k,a}(\gamma)f$$

is continuous.

(2) *For any $p \in \mathscr{H}_k^{(m)}(\mathbb{R}^N)$ and $\ell \in \mathbb{N}$, $\Phi_\ell^{(a)}(p, \cdot)$ (see* (3.28)*) is an eigenfunction of the operator* $\Omega_{k,a}(\gamma_z) = \exp(\omega_{k,a}(-z\mathbf{k}))$*:*

$$\Omega_{k,a}(\gamma_z)\Phi_\ell^{(a)}(p, x) = e^{-z(\lambda_{k,a,m}+1+2\ell)}\Phi_\ell^{(a)}(p, x),$$

where $\lambda_{k,a,m} = \frac{1}{a}(2m + 2\langle k\rangle + N - 2)$ *(see* (3.11)*).*

(3) *The operator norm* $\|\Omega_{k,a}(\gamma_z)\|_{\mathrm{op}}$ *is* $\exp(-\frac{1}{a}(2\langle k\rangle + N + a - 2)\operatorname{Re} z)$.

(4) *If* $\operatorname{Re}(z) > 0$*, then $\Omega_{k,a}(\gamma_z)$ is a Hilbert–Schmidt operator.*

(5) *If* $\operatorname{Re}(z) = 0$*, then $\Omega_{k,a}(\gamma_z)$ is a unitary operator.*

(6) *The representation $\Omega_{k,a}$ is faithful on $\widetilde{\Gamma(W)}$ if at least one of a or $\langle k\rangle$ is irrational, and on $\widetilde{\Gamma(W)}/D$ for some discrete abelian kernel D if both a and $\langle k\rangle$ are rational.*

Proof. The second statement follows from (3.33 a). The fifth statement is a special case of Theorem 3.30. The proof of the other statements is parallel to that of [36, Proposition 3.6.1], and we omit it. □

4. The integral representation of the holomorphic semigroup $\Omega_{k,a}(\gamma_z)$

We have seen in Theorem 3.39 that $\Omega_{k,a}(\gamma_z)$ is a Hilbert–Schmidt operator for $\operatorname{Re} z > 0$ and is a unitary operator for $\operatorname{Re} z = 0$. By the Schwartz kernel theorem, the operator $\Omega_{k,a}(\gamma_z)$ can be expressed by means of a distribution kernel $\Lambda_{k,a}(x, y; z)$. If we adopt Gelfand's notation on a generalized functions, we may write the operator $\Omega_{k,a}(\gamma_z)$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ as an 'integral transform' against the measure $\vartheta_{k,a}(x)dx$:

$$\Omega_{k,a}(\gamma_z)f(x) = c_{k,a}\int_{\mathbb{R}^N} \Lambda_{k,a}(x, y; z) f(y)\vartheta_{k,a}(y)dy. \tag{4.1}$$

Here, we have normalized the kernel $\Lambda_{k,a}(x, y; z)$ by the constant $c_{k,a}$ that will be defined in (4.47). In light of the unitary isomorphism

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \xrightarrow{\sim} L^2(\mathbb{R}^N, dx), \ \ f(x) \mapsto f(x)\vartheta_{k,a}(x)^{\frac{1}{2}}.$$

We see that $\Lambda_{k,a}(x, y; z)\vartheta_{k,a}(x)^{\frac{1}{2}}\vartheta_{k,a}(y)^{\frac{1}{2}}$ is a tempered distribution of $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

The goal of this section is to find the kernel $\Lambda_{k,a}(x, y; z)$. The main result of this section is Theorem 4.23.

4.1. **Integral representation for the radial part of $\Omega_{k,a}(\gamma_z)$.**
By Lemma 3.7, the $\mathfrak{sl}_2$-action $\omega_{k,a}$ on $C^\infty(\mathbb{R}^N \setminus \{0\})$ (see (3.5) for definition) can be described in a simple form on each k-spherical component $\mathscr{H}_k^m(\mathbb{R}^N)$, namely, it can be expressed as the action only on the radial direction. Accordingly, we can define the 'radial part' of the holomorphic semigroup $\Omega^{(m)}_{k,a}(\gamma_z)$ (see (4.4) below for definition) on $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$. The main result of this subsection is the integral formula for $\Omega^{(m)}_{k,a}(\gamma_z)$, which will be given in Theorems 4.4 and 4.5.

4.1.1. *Radial part of holomorphic semigroup.*
Recall that $\mathscr{H}_k^m(\mathbb{R}^N)$ is the space of k-harmonic polynomials of degree $m \in \mathbb{N}$. Let

$$\alpha^{(m)}_{k,a} : \mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}} \otimes L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr) \longrightarrow L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$$

be a linear map defined by

$$\alpha^{(m)}_{k,a}(p \otimes f)(x) = p(\frac{x}{||x||})f(||x||) \quad \text{for } p \in \mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}} \text{ and } f \in L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr).$$

Summing up $\alpha^{(m)}_{k,a}$, we get a direct sum decomposition of the Hilbert space:

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) = \sum_{m\in\mathbb{N}}^{\oplus} \mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}} \otimes L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr). \tag{4.2}$$

It follows from Theorem 3.31 that the unitary representation $\Omega_{k,a}$ of $\widetilde{SL(2,\mathbb{R})}$ on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ induces a family of unitary operators, to be denoted by $\Omega^{(m)}_{k,a}(\gamma_z)$ ($z \in i\mathbb{R}, m \in \mathbb{N}$), on $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$ such that

$$\alpha^{(m)}_{k,a}\Big(p \otimes \Omega^{(m)}_{k,a}(\gamma_z)(f)\Big) = \Omega_{k,a}(\gamma_z)\Big(\alpha^{(m)}_{k,a}(p \otimes f)\Big). \tag{4.3}$$

As is Theorem 3.39 for $\Omega_{k,a}(\gamma_z)$, the unitary operator $\Omega^{(m)}_{k,a}(\gamma_z)$ extends to a holomorphic semigroup of Hilbert–Schmidt operators on $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$ for $\mathrm{Re}(z) > 0$. Further, there exists a unique kernel $\Lambda^{(m)}_{k,a}(r, s; z)$ for each z and $m \in \mathbb{N}$ such that

$$\Omega^{(m)}_{k,a}(\gamma_z)f(r) = \int_0^\infty f(s)\Lambda^{(m)}_{k,a}(r, s; z)s^{2\langle k\rangle+N+a-3}ds, \tag{4.4}$$

holds for any $f \in L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$.

According to the direct sum (4.2), the semigroup $\Omega_{k,a}(\gamma_z)$ is decomposed as follows:

$$\Omega_{k,a}(\gamma_z) = \sum_{m\in\mathbb{N}}^{\oplus} \mathrm{id}_{|\mathscr{H}_k^m(\mathbb{R}^N)} \otimes \Omega^{(m)}_{k,a}(\gamma_z). \tag{4.5}$$

Comparing the integral expressions (4.1) and (4.4) of $\Omega_{k,a}(\gamma_z)$ and $\Omega^{(m)}_{k,a}(\gamma_z)$ respectively, we see that the kernels $\Lambda_{k,a}(x, y; z)$ and $\Lambda^{(m)}_{k,a}(r, s; z)$ satisfy the following identities:

$$c_{k,a}\int_{\mathbb{R}^N} \Lambda_{k,a}(x, y; z)p\Big(\frac{y}{||y||}\Big)f(||y||)\vartheta_{k,a}(y)dy = p\Big(\frac{x}{||x||}\Big)\int_0^\infty f(s)\Lambda^{(m)}_{k,a}(r, s; z)s^{2\langle k\rangle+N+a-3}ds \tag{4.6}$$

for any $p \in \mathscr{H}_k^{(m)}(\mathbb{R}^N)$ and $f \in L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$.

In light of the following formula for the measures

$$\vartheta_{k,a}(y)dy = \vartheta_k(\eta)s^{2\langle k\rangle+N+a-3}d\sigma(\eta)ds$$

with respect to polar coordinates $y = s\eta$, we see that (4.6) is equivalent to

$$c_{k,a}\int_{S^{N-1}}\Lambda_{k,a}(r\omega, s\eta; z)p(\eta)\vartheta_k(\eta)d\sigma(\eta) = p(\omega)\Lambda_{k,a}^{(m)}(r,s;z). \tag{4.7}$$

Therefore, the distribution $\Lambda_{k,a}$ is determined by the set of functions $\Lambda_{k,a}^{(m)}$ ($m \in \mathbb{N}$) as follows:

Proposition 4.1. *Fix $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 0$. Then, the distribution $\Lambda_{k,a}(x,y;z)$ on $\mathbb{R}^N \times \mathbb{R}^N$ is characterized by the condition (4.7) for any $p \in \mathscr{H}_k^m(\mathbb{R}^N)$ and any $m \in \mathbb{N}$.*

The relation between $\Lambda_{k,a}$ and $\Lambda_{k,a}^{(m)}$ ($m \in \mathbb{N}$) will be discussed again in Theorem 4.20 by means of the 'Poisson kernel'.

4.1.2. *The case* $\operatorname{Re}(z) > 0$.
Suppose $\operatorname{Re}(z) > 0$. Then, $\Omega_{k,a}^{(m)}(\gamma_z)$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$, and consequently, the kernel $\Lambda_{k,a}^{(m)}(\cdot,\cdot\ ;z)$ is square integrable function with respect to the measure $(rs)^{2\langle k\rangle+N+a-3}dr\,ds$.

We shall find a closed formula for $\Lambda_{k,a}^{(m)}(r,s;z)$. Let us fix $m \in \mathbb{N}$ (as well as k and a) once and for all. We have given in Proposition 3.15 an explicit orthonormal basis $\{f_{\ell,m}^{(a)}(r) : \ell \in \mathbb{N}\}$ of $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$. On the other hand, it follows from Theorem 3.39 (2) that $\widetilde{\Phi}_\ell^{(a)}(p,x) = f_{\ell,m}^{(a)}(r)p(\omega)$ (see (3.35)) is an eigenfunction of the Hilbert–Schmidt operator $\Omega_{k,a}(\gamma_z)$:

$$\Omega_{k,a}(\gamma_z)\widetilde{\Phi}_\ell^{(a)}(p,x) = e^{-z(2\ell+\lambda_{k,a,m}+1)}\widetilde{\Phi}_\ell^{(a)}(p,x).$$

Using the identity (4.3), we deduce that

$$\Omega_{k,a}^{(m)}(\gamma_z)f_{\ell,m}^{(a)}(r) = e^{-z(2\ell+\lambda_{k,a,m}+1)}f_{\ell,m}^{(a)}(r), \tag{4.8}$$

where the constant $\lambda_{k,a,m}$ is defined in (3.11). Hence, the kernel $\Lambda_{k,a}^{(m)}(r,s;z)$ in (4.4) is given by the following series expansion:

$$\Lambda_{k,a}^{(m)}(r,s;z) = \sum_{\ell=0}^{\infty} f_{\ell,m}^{(a)}(r)f_{\ell,m}^{(a)}(s)e^{-z(\lambda_{k,a,m}+1+2\ell)}.$$

In view of the definition (3.32) of $f_{\ell,m}^{(a)}(r)$, $\Lambda_{k,a}^{(m)}(r,s;z)$ amounts to

$$\frac{e^{-z(\lambda_{k,a,m}+1)}(rs)^m e^{-\frac{1}{a}(r^a+s^a)}}{a^{\lambda_{k,a,m}}2^{-(\lambda_{k,a,m}+1)}}\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+1)}{\Gamma(\lambda_{k,a,m}+\ell+1)}e^{-2\ell z}L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a}r^a\Big)L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a}s^a\Big).$$

In order to compute this series expansion, we recall some basic identities of Bessel functions. Let I_λ be the I-Bessel function defined by

$$I_\lambda(w) := e^{-\frac{\pi}{2}\lambda i}J_\lambda(e^{\frac{\pi}{2}i}w).$$

It is also convenient to introduce the normalized I-Bessel function by

$$\widetilde{I}_\lambda(w) := \Big(\frac{w}{2}\Big)^{-\lambda} I_\lambda(w) = \sum_{\ell=0}^{\infty} \frac{w^{2\ell}}{2^{2\ell}\ell!\Gamma(\lambda+\ell+1)} \tag{4.9}$$

$$= \frac{1}{\sqrt{\pi}\,\Gamma(\lambda+\frac{1}{2})} \int_{-1}^{1} e^{wt}(1-t^2)^{\lambda-\frac{1}{2}} dt. \tag{4.10}$$

We note that $\widetilde{I}_\lambda(w)$ is an entire function of $w \in \mathbb{C}$ satisfying

$$\widetilde{I}_\lambda(0) = \frac{1}{\sqrt{\pi}\,\Gamma(\lambda+\frac{1}{2})}.$$

Now, we can use the following Hille–Hardy identity [1, (6.2.25)]

$$\sum_{\kappa=0}^{\infty} \frac{\Gamma(\kappa+1)}{\Gamma(\lambda+\kappa+1)} L_\kappa^{(\lambda)}(u) L_\kappa^{(\lambda)}(v) w^\kappa = \frac{1}{1-w} \exp\Big(-\frac{(u+v)w}{1-w}\Big)(uvw)^{-\frac{\lambda}{2}} I_\lambda\Big(\frac{2\sqrt{uvw}}{1-w}\Big)$$
$$= \frac{1}{(1-w)^{\lambda+1}} \exp\left(-\frac{(u+v)w}{1-w}\right) \widetilde{I}_\lambda\left(\frac{2\sqrt{uvw}}{1-w}\right).$$

Here the left-hand side converges for $|w| < 1$. Hence, we get a closed formula for $\Lambda_{k,a}^{(m)}(r,s;z)$:

$$\Lambda_{k,a}^{(m)}(r,s;z) = \frac{(rs)^{-\langle k\rangle - \frac{N}{2}+1}}{\sinh(z)} e^{-\frac{1}{a}(r^a+s^a)\coth(z)} I_{\lambda_{k,a,m}}\left(\frac{2}{a}\frac{(rs)^{\frac{a}{2}}}{\sinh(z)}\right) \tag{4.11}$$
$$= \frac{(rs)^m}{a^{\lambda_{k,a,m}}(\sinh(z))^{\lambda_{k,a,m}+1}} e^{-\frac{1}{a}(r^a+s^a)\coth(z)} \widetilde{I}_{\lambda_{k,a,m}}\left(\frac{2}{a}\frac{(rs)^{\frac{a}{2}}}{\sinh(z)}\right).$$

Next, let us give an upper estimate of the kernel function $\Lambda_{k,a}^{(m)}(r,s;z)$. For this, we recall from [36, §4.2] the following elementary lemma.

Lemma 4.2. *For $z = x+iy$, we set*

$$\alpha(z) := \frac{\sinh(2x)}{\cosh(2x)-\cos(2y)},$$
$$\beta(z) := \frac{\cos(y)}{\cosh(x)}.$$

Then, we have

1)

$$\operatorname{Re}\coth(z) = \alpha(z), \tag{4.12}$$

$$\operatorname{Re}\frac{1}{\sinh(z)} = \alpha(z)\beta(z). \tag{4.13}$$

2) *If $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$, then we have* $\cosh(2x)-\cos(2y) > 0$, *and*

$$\alpha(z) \geq 0 \quad \textit{and} \quad |\beta(z)| < 1.$$

3) *If* $\operatorname{Re} z > 0$, *then* $\alpha(z) > 0$.

We set

$$C(k,a,m;z) := \frac{1}{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m}+1)\,|\sinh(z)|^{\lambda_{k,a,m}+1}}. \tag{4.14}$$

With these notations, we have:

Lemma 4.3. *For $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$, the kernel function $\Lambda^{(m)}_{k,a}(r,s;z)$ has the following upper estimate:*

$$|\Lambda^{(m)}_{k,a}(r,s;z)| \leq C(k,a,m;z)(rs)^m \exp\Big(-\frac{1}{a}(r^a+s^a)\alpha(z)(1-|\beta(z)|)\Big). \tag{4.15}$$

Proof. By the following upper estimate of the I-Bessel function (see [36, Lemma 8.5.1])

$$|\widetilde{I}_\nu(w)| \leq \Gamma(\nu+1)^{-1}e^{|\operatorname{Re}(w)|} \quad \text{for } \nu \geq -\frac{1}{2} \text{ and } w \in \mathbb{C} \tag{4.16}$$

that we get

$$|\Lambda^{(m)}_{k,a}(r,s;z)| \leq C(k,a,m;z)(rs)^m \exp\Big(-\frac{1}{a}(r^a+s^a)\Big(\operatorname{Re}\coth(z) - \Big|\operatorname{Re}\frac{1}{\sinh(z)}\Big|\Big)\Big). \tag{4.17}$$

Here, we have used $r^a + s^a \geq 2(rs)^{\frac{a}{2}}$. Then, the substitution of (4.12) and (4.13) shows Lemma. □

We are ready to complete the proof of the following:

Theorem 4.4. *Let $\gamma_z = \operatorname{Exp}\Big(iz\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Big)$ be an element of $\widetilde{\Gamma(W)}$ (see (3.44)), and $\Lambda^{(m)}_{k,a}(r,s;z)$ the function defined by (4.11). Assume $m \in \mathbb{N}$ and $a > 0$ satisfy*

$$2m + 2\langle k\rangle + N + a - 2 > 0. \tag{4.18}$$

Then, for $z \in \mathbb{C}^{++}$, the Hilbert–Schmidt operator $\Omega^{(m)}_{k,a}(\gamma_z)$ on $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$ is given by

$$\Omega^{(m)}_{k,a}(\gamma_z)f(r) = \int_0^\infty \Lambda^{(m)}_{k,a}(r,s;z)f(s)s^{2\langle k\rangle+N+a-3}ds. \tag{4.19}$$

The integral in (4.19) converges absolutely for $f \in L^2(\mathbb{R}_+, s^{2\langle k\rangle+N+a-3}ds)$.

Proof. We have already proved the formula (4.11) for $\Lambda^{(m)}_{k,a}(r,s;z)$. The convergence of the integral (4.19) is deduced from the Cauchy–Schwarz inequality because $\Lambda^{(m)}_{k,a}(r,\cdot\,;z) \in L^2(\mathbb{R}_+, s^{2\langle k\rangle+N+a-3}ds)$ for all $z \in \mathbb{C}^{++}$ if (4.18) is fulfilled. □

4.1.3. *The case* $\operatorname{Re}(z) = 0$.
The operator $\Omega^{(m)}_{k,a}(\gamma_z)$ is unitary if $\operatorname{Re}(z) = 0$. In this subsection, we discuss its distribution kernel.

We note that the substitution of $z = i\mu$ into (4.11) makes sense as far as $\mu \notin \pi\mathbb{Z}$, and we get the following formula

$$\Lambda^{(m)}_{k,a}(r,s;i\mu) = \exp\Big(-\frac{\pi i}{2}(\lambda_{k,a,m}+1)\Big)\frac{(rs)^{-\langle k\rangle-\frac{N}{2}+1}}{\sin\mu}\exp\Big(\frac{i}{a}(r^a+s^a)\cot(\mu)\Big)J_{\lambda_{k,a,m}}\left(\frac{2}{a}\frac{(rs)^{\frac{a}{2}}}{\sin\mu}\right). \tag{4.20}$$

Here, we have used the relation $I_\lambda\big(\frac{z}{i}\big) = e^{-\frac{i\lambda\pi}{2}}J_\lambda(z)$.

In this subsection, we shall prove:

Theorem 4.5. *Retain the notation and the assumption* (4.18) *as in Theorem 4.4. For* $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$, *the unitary operator* $\Omega^{(m)}_{k,a}(\gamma_{i\mu})$ *on* $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$ *is given by*

$$\Omega^{(m)}_{k,a}(\gamma_{i\mu})f(r) = \int_0^\infty f(s)\Lambda^{(m)}_{k,a}(r,s;i\mu)s^{2\langle k\rangle+N+a-3}ds. \tag{4.21}$$

The integral in the right-hand side (4.21) *converges absolutely for all* f *in the dense subspace, in* $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$, *spanned by the functions* $\{f^{(a)}_{\ell,m}\}_{\ell\in\mathbb{N}}$ *(see* (3.32) *for definition).*

Proof. Let $\epsilon > 0$ and $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$. By Theorem 4.4 we have

$$\Omega^{(m)}_{k,a}(\gamma_{\epsilon+i\mu})f(r) = \int_0^\infty f(s)\Lambda^{(m)}_{k,a}(r,s;\epsilon+i\mu)s^{2\langle k\rangle+N+a-3}ds. \tag{4.22}$$

As $\epsilon \to 0$ the left-hand side converges to $\Omega^{(m)}_{k,a}(\gamma_{i\mu})$ by Theorem 3.39 (1).

On the other hand, the addition formula

$$\operatorname{csch}(\epsilon+i\mu) = \frac{\operatorname{csch}(\epsilon)\operatorname{csch}(i\mu)}{\coth(\epsilon)+\coth(i\mu)}$$

gives

$$|\operatorname{csch}(\epsilon+i\mu)| < |\operatorname{csch}(i\mu)|. \tag{4.23}$$

Hence, it follows from Lemma 4.3 that we have

$$|\Lambda^{(m)}_{k,a}(r,s;\epsilon+i\mu)| \le C(k,a,m;i\mu)(rs)^m \le C\frac{(rs)^m}{|\sin(\mu)|^{\lambda_{k,a,m}+1}},$$

for some constant C. In view of (3.32), we get

$$\left|\Lambda^{(m)}_{k,a}(r,s;\epsilon+i\mu)f^{(a)}_{\ell,m}(s)\right| \le C'\frac{r^m\exp(-\frac{1}{a}s^a)}{|\sin(\mu)|^{\lambda_{k,a,m}+1}}s^{2m}\left|L^{(\lambda_{k,a,m})}_\ell\Big(\frac{2}{a}s^a\Big)\right|.$$

Now, we can use the dominated convergence theorem to deduce that the right-hand side of (4.22) goes to

$$\int_0^\infty f(s)\Lambda^{(m)}_{k,a}(r,s;i\mu)s^{2\langle k\rangle+N+a-3}ds$$

as $\epsilon \to 0$. Hence, Theorem has been proved. □

As a corollary of Theorem 4.5, we obtain representation theoretic proofs of the following two classical integral formulas of Bessel functions:

Corollary 4.6.

(1) *(Weber's second exponential integral,* [22, 6.615]*).*

$$\int_0^\infty e^{-\delta T}J_\nu(2\alpha\sqrt{T})J_\nu(2\beta\sqrt{T})dT = \frac{1}{\delta}e^{-\frac{1}{\delta}(\alpha^2+\beta^2)}I_\nu\Big(\frac{2\alpha\beta}{\delta}\Big),$$

where $|\arg(\delta)| < \frac{\pi}{2}$ *and* $\nu \ge 0$.

(2) *(See* [22, 7.421.4]*).*

$$\int_0^\infty e^{-\delta T}L^{(\nu)}_\ell(\alpha T)J_\nu(\beta\sqrt{T})T^{\frac{\nu}{2}}dT = \frac{(\delta-\alpha)^\ell\beta^\nu}{2^\nu\delta^{\nu+\ell+1}}e^{-\frac{\beta^2}{4\delta}}L^{(\nu)}_\ell\Big(\frac{\alpha\beta^2}{4\delta(\alpha-\delta)}\Big), \tag{4.24}$$

for $\operatorname{Re}(\delta) > 0$ *and* $\operatorname{Re}(\nu) > 0$.

Proof (Sketch). (1) The semigroup law $\Omega_{k,a}^{(m)}(\gamma_{z_1})\Omega_{k,a}^{(m)}(\gamma_{z_2}) = \Omega_{k,a}^{(m)}(\gamma_{z_1+z_2})$ yields

$$\int_0^\infty \Lambda_{k,a}^{(m)}(r, s; z_1)\Lambda_{k,a}^{(m)}(s, r'; z_2)s^{2\langle k\rangle+N+a-3}ds = \Lambda_{k,a}^{(m)}(r, r'; z_1 + z_2). \tag{4.25}$$

Using the expression (4.20) of $\Lambda_{k,a}^{(m)}$, we get the identity (1).

(2) The identity (4.8) (in terms of group theory, this comes from the K-type formula (3.34 a)) can be restated as

$$\int_0^\infty \Lambda_{k,a}^{(m)}(r, s; z) f_{\ell,m}^{(a)}(s)s^{2\langle k\rangle+N+a-3}ds = e^{-z(2\ell+\lambda_{k,a,m}+1)} f_{\ell,m}^{(a)}(r),$$

in terms of the integral kernels by Theorem 4.4. After some simplifications and by putting constants together, we get the identity (4.24). □

Remark 4.7. 1) *The operator* $\pi(\lambda_{k,a,m})(\gamma_z)$ *acts on the irreducible representation* $\pi(\lambda_{k,a,m})$ *of* $\widetilde{SL(2,\mathbb{R})}$ *as a scalar multiplication if* $z \in i\pi\mathbb{Z}$ *i.e. if* γ_z *belongs to the center (see Fact 3.27). Correspondingly, the kernel function* $\Lambda_{k,a}^{(m)}(r, s; \epsilon + i\mu)$ *approaches to a scalar multiple of Dirac's delta function as* ϵ *goes to* 0 *if* $\mu \in \pi\mathbb{Z}$.

2) *Of particular interest is another case where* $\mu \in \pi(\mathbb{Z} + \frac{1}{2})$. *For simplicity, let* $\mu = \frac{\pi}{2}$. *Then, the formula* (4.20) *collapses to*

$$\Lambda_{k,a}^{(m)}\Big(r, s; \frac{\pi i}{2}\Big) = \exp\Big(-\frac{\pi i}{2}(\lambda_{k,a,m} + 1)\Big)(rs)^{-\langle k\rangle-\frac{N}{2}+1} J_{\lambda_{k,a,m}}\Big(\frac{2}{a}(rs)^{\frac{a}{2}}\Big).$$

For $a = 1, 2$, *we have*

$$\Lambda_{k,1}^{(m)}(r, s; \frac{\pi i}{2}) = e^{-i\frac{\pi}{2}(2m+2\langle k\rangle+N-1)}(rs)^{-\langle k\rangle-\frac{N}{2}+1} J_{2m+2\langle k\rangle+N-2}(2\sqrt{rs}) \quad (a = 1),$$

$$\Lambda_{k,2}^{(m)}(r, s; \frac{\pi i}{2}) = e^{-i\frac{\pi}{2}(m+\langle k\rangle+\frac{N}{2})}(rs)^{-\langle k\rangle-\frac{N}{2}+1} J_{m+\langle k\rangle+\frac{N}{2}-1}(rs) \quad (a = 2).$$

We shall discuss the unitary operator $\Omega(\gamma_{\frac{\pi i}{2}}) = \lim_{\epsilon\to 0} \Omega(\gamma_{\frac{\epsilon+\pi i}{2}})$ in full detail, which we call the (k, a)-*generalized Fourier transform* $\mathscr{F}_{k,a}$ (up to a phase factor) in Section 5.

4.2. **Gegenbauer transform.**

In this section, we summarize some basic properties of the Gegenbauer polynomials and the corresponding integral transforms.

4.2.1. *The Gegenbauer polynomial.*

The Gegenbauer polynomial $C_m^\nu(t)$ of degree m is defined by the generating function

$$(1 - 2rt + r^2)^{-\nu} = \sum_{m=0}^{\infty} C_m^\nu(t)r^m. \tag{4.26}$$

To be more explicit, it is given as

$$C_m^\nu(t) = (- \frac{1}{2})^m \frac{(2\nu)_m}{m!(\nu + \frac{1}{2})_m}(1 - t^2)^{-\nu+\frac{1}{2}} \frac{d^m}{dt^m}(1 - t^2)^{m+\nu-\frac{1}{2}}. \tag{4.27}$$

If we put $t = \cos\theta$, and expand

$$(1 - 2r\cos\theta + r^2)^{-\nu} = ((1 - re^{i\theta})(1 - re^{-i\theta}))^{-\nu}$$

by the binomial theorem, then we have

$$C_m^\nu(\cos\theta) = \sum_{k=0}^{m} \frac{(\nu)_k(\nu)_{m-k}}{k!(m-k)!}\cos(m-2k)\theta. \tag{4.28}$$

Then, the following fact is readily seen.

Fact 4.8. 1) *$C_m^\nu(t)$ is a polynomial of t of degree m, and is also a polynomial in parameter ν.*
2) *$C_m^0(t) \equiv 0$ for any $m \geq 1$.*
3) *$C_0^\nu(t) \equiv 1$, $C_1^\nu(t) = 2\nu t$.*
4) *$C_m^\nu(1) = \dfrac{\Gamma(m+2\nu)}{m!\Gamma(2\nu)}$.*

In this subsection, we prove:

Lemma 4.9. *Fix $\nu \in \mathbb{R}$. Then there exists a constant $B(\nu) > 0$ such that*

$$\sup_{-1\leq t\leq 1}\left|\frac{1}{\nu}C_m^\nu(t)\right| \leq B(\nu)m^{2\nu-1} \quad \textit{for any } m \in \mathbb{N}_+. \tag{4.29}$$

Remark 4.10. 1) *For $\nu > 0$, it is known that the upper bound of $|C_m^\nu(t)|$ is attained at $t = 1$, namely,*

$$\sup_{-1\leq t\leq 1}\left|C_m^\nu(t)\right| = C_m^\nu(1) = \frac{\Gamma(m+2\nu)}{m!\Gamma(2\nu)}.$$

This can be verified easily by (4.28)*, or alternatively, by the following integral expression for $\nu > 0$:*

$$C_m^\nu(t) = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(m+2\nu)}{\sqrt{\pi}m!\Gamma(\nu)\Gamma(2\nu)}\int_0^\pi (t+\sqrt{t^2-1}\cos\theta)^m \sin^{2\nu-1}\theta\, d\theta. \tag{4.30}$$

2) *For $\nu = 0$, the left-hand side of* (4.29) *is interpreted as the L^∞-norm of* $\lim_{\nu\to 0}\frac{1}{\nu}C_m^\nu(t)$, *which is a polynomial of t by Fact 4.8 2).*
3) *Our proof below works also for all $\nu \in \mathbb{C}$.*

Before proving Lemma 4.9, we prepare the following estimate:

Claim 4.11. *Let $\lambda \in \mathbb{R}$. Then there exists a constant $A(\lambda) > 0$ such that*

$$\left|\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)k!}\right| \leq A(\lambda)k^{\lambda-1} \quad \textit{for any } k \in \mathbb{N}.$$

Proof. We recall Stirling's asymptotic formula of the Gamma function:

$$\Gamma(x) \sim \Gamma_0(x) \quad \text{as } x \to \infty,$$

where $\Gamma_0(x) := \sqrt{2\pi}x^{x-1}e^{-x}$. In light of the following ratio:

$$\frac{\Gamma_0(k+\lambda)}{\Gamma_0(k+1)} = k^{\lambda-1}\left(\frac{1+\frac{\lambda}{k}}{1+\frac{1}{k}}\right)^{k+\frac{1}{2}}\left(1+\frac{\lambda}{k}\right)^{\lambda-1} e^{1-\lambda},$$

we get

$$\lim_{k\to\infty}\frac{\Gamma(\lambda+k)}{k!k^{\lambda-1}} = 1.$$

Thus, Claim follows. $\square$

Proof of Lemma 4.9. By (4.28), we have

$$\sup_{-1\le t\le 1}|C_m^\nu(t)| \le \sum_{k=0}^{m}\left|\frac{\Gamma(\nu+k)}{k!\Gamma(\nu)}\frac{\Gamma(m-k+\nu)}{(m-k)!\Gamma(\nu)}\right|.$$

We note that there is no pole in the Gamma factors in the right-hand side. We now use Claim 4.11, and get

$$\le \sum_{k=0}^{m} A(\nu)^2 k^{\nu-1}(m-k)^{\nu-1}$$
$$\le A(\nu)^2(m+1)\left(\frac{m}{2}\right)^{2\nu-2}.$$

Hence, (4.29) is proved for $\nu \neq 0$.

For $\nu = 0$, we use (4.28), and get

$$\lim_{\nu\to 0}\frac{1}{\nu}C_m^\nu(\cos\theta) = \frac{1}{m}(\cos(n-2m)\theta + \cos n\theta). \tag{4.31}$$

Hence, the inequality (4.29) also holds for $\nu = 0$. Thus, we have proved Lemma 4.9. □

4.2.2. *The Gegenbauer transform.*

We summarize L^2-properties of Gegenbauer polynomials in a way that we shall use later.

Fact 4.12 (see [1], [7, Chapter 15])**.** *Suppose* $\nu > -\frac{1}{2}$.

1) $\{C_m^\nu(t) : m \in \mathbb{N}\}$ *is an orthogonal basis in the Hilbert space* $\mathscr{H}_\nu := L^2((-1,1),(1-t^2)^{\nu-\frac{1}{2}}dt)$.

2) $\displaystyle\int_{-1}^{1} C_m^\nu(t)^2(1-t^2)^{\nu-\frac{1}{2}}dt = \frac{\pi\Gamma(2\nu+m)}{2^{2\nu-1}\Gamma(m+1)(m+\nu)\Gamma(\nu)^2}$.

3) *We set a normalized constant* $b_{\nu,m}$ *by*

$$b_{\nu,m} := \frac{2^{2\nu-1}\Gamma(m+1)\Gamma(\nu)\Gamma(\nu+1)}{\pi\Gamma(m+2\nu)}. \tag{4.32}$$

Then, the Gegenbauer transform defined by

$$\mathscr{C}_{\nu,m}(h) := b_{\nu,m}\int_{-1}^{1} h(t)C_m^\nu(t)(1-t^2)^{\nu-\frac{1}{2}}dt, \quad \textit{for } h \in \mathscr{H}_\nu \tag{4.33}$$

has the following inversion formula.

$$h = \frac{1}{\nu}\sum_{m=0}^{\infty}(m+\nu)\mathscr{C}_{\nu,m}(h)C_m^\nu(t). \tag{4.34}$$

The orthonality relation in Fact 4.12 can be restated in terms of the Gegenbauer transform as follows:

$$\mathscr{C}_{\nu,m}(C_n^\nu) = \begin{cases}\dfrac{\nu}{m+\nu} & (n=m),\\ 0 & (n\neq m).\end{cases} \tag{4.35}$$

The Gegenbauer transform $\mathscr{C}_{\nu,m}$ arises also in a Dunkl analogue of the classical Funk–Hecke formula for spherical harmonics as follows.

Fact 4.13 (see [61, Theorem 2.1]). *Let h be a continuous function on* $[-1,1]$ *and* $p \in \mathcal{H}_k^m(\mathbb{R}^N)$. *Then, we have*

$$d_k \int_{S^{N-1}} (\widetilde{V}_k h)(\omega,\eta) p(\eta) \vartheta_k(\eta) d\sigma(\eta) = \mathscr{C}_{\langle k\rangle + \frac{N-2}{2}, m}(h) p(\omega), \qquad \omega \in S^{N-1}. \tag{4.36}$$

Here, d_k is the constant defined in (2.12)*, and $\widetilde{V}_k h$ is defined in* (2.6) *by using the Dunkl intertwining operator V_k. For $k \equiv 0$, $(\widetilde{V}_k h)(\eta,\omega) = h(\langle \eta,\omega\rangle)$ and $\vartheta_k(\eta) \equiv 1$, so that the identity* (4.36) *collapses to the original Funk–Hecke formula.*

4.2.3. *Explicit formulas of Gegenbauer transforms.*
In this subsection, we present two explicit formulas of Gegenbauer transforms $\mathscr{C}_{\nu,m}$ (see (4.33)). These results will be used in describing the kernel distributions $\Lambda_{k,a}(x,y;z)$ for $a = 1, 2$ (see Theorem 4.24).

Lemma 4.14.
1) $\mathscr{C}_{\nu,m}\Big(\widetilde{I}_{\nu-\frac{1}{2}}\Big(\alpha(1+t)^{\frac{1}{2}}\Big)\Big) = 2^{2\nu-m}\pi^{-\frac{1}{2}}\alpha^{2m}\Gamma(\nu+1)\widetilde{I}_{2m+2\nu}(\sqrt{2}\,\alpha) = \frac{\alpha^{2m}\Gamma(2\nu+1)}{2^m\Gamma(\nu+\frac{1}{2})}\widetilde{I}_{2m+2\nu}(\sqrt{2}\alpha).$
2) $\mathscr{C}_{\nu,m}(e^{\alpha t}) = 2^{-m}\alpha^m\Gamma(\nu+1)\widetilde{I}_{\nu+m}(\alpha).$

This lemma is an immediate consequence of the following integral formulas and the duplication formula of the Gamma function:

$$\Gamma(2\nu) = 2^{2\nu-1}\pi^{-\frac{1}{2}}\Gamma(\nu)\Gamma\Big(\nu+\frac{1}{2}\Big). \tag{4.37}$$

Lemma 4.15. *For $\alpha, \nu \in \mathbb{C}$ such that* $\mathrm{Re}(\nu) > 0$, *the following two integral formulas hold:*

(1)

$$\int_{-1}^{1} \widetilde{I}_{\nu-\frac{1}{2}}(\alpha(1+t)^{\frac{1}{2}}) C_m^{\nu}(t)(1-t^2)^{\nu-\frac{1}{2}} dt = \frac{2\sqrt{\pi}\Gamma(m+2\nu)}{\Gamma(m+1)\Gamma(\nu)}\Big(\frac{\alpha}{\sqrt{2}}\Big)^{2m}\widetilde{I}_{2m+2\nu}(\sqrt{2}\alpha). \tag{4.38}$$

(2)

$$\int_{-1}^{1} e^{\alpha t} C_m^{\nu}(t)(1-t^2)^{\nu-\frac{1}{2}} dt = \frac{2^{-2\nu-m+1}\pi\Gamma(2\nu+m)}{\Gamma(m+1)\Gamma(\nu)}\alpha^m \widetilde{I}_{\nu+m}(\alpha). \tag{4.39}$$

Proof of Lemma 4.15. (1) This identity was proved in [36, Lemma 8.5.2].

(2) The integral formula (4.39) is well known (see for instance [58, page 570]). However, for the convenience of the readers, we give a simple proof. Using (4.27), we have

$$\begin{aligned} \int_{-1}^{1} e^{\alpha t} C_m^{\nu}(t)(1-t^2)^{\nu-\frac{1}{2}} dt &= \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)} \frac{\Gamma(m+2\nu)}{2^m m!\Gamma(m+\nu+\frac{1}{2})} \int_{-1}^{1} \frac{d^m}{dt^m}(e^{\alpha t})(1-t^2)^{m+\nu-\frac{1}{2}} dt \\ &= \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)} \frac{\Gamma(m+2\nu)}{2^m m!\Gamma(m+\nu+\frac{1}{2})} \alpha^m \int_{-1}^{1} e^{\alpha t}(1-t^2)^{m+\nu-\frac{1}{2}} dt. \end{aligned}$$

Now, (4.39) follows from the integral representation (4.10) of $\widetilde{I}_\nu(w)$. □

Remark 4.16 (expansion formula)**.** *Applying the inversion formula of the Gegenbauer transform (see Fact 4.12), we get the following expansion formulas from Lemma 4.14:*

$$e^{wt} = \Gamma(\nu)\sum_{m=0}^{\infty}(\nu+m)\Big(\frac{w}{2}\Big)^{m}\widetilde{I}_{\nu+m}(w)C_m^{\nu}(t), \qquad \mathrm{Re}(\nu)>0 \tag{4.40}$$

$$\widetilde{I}_{\nu-1/2}\Big(\frac{w(1+t)^{1/2}}{\sqrt{2}}\Big) = \frac{2^{2\nu}\Gamma(\nu)}{\sqrt{\pi}}\sum_{m=0}^{\infty}(\nu+m)\Big(\frac{w}{2}\Big)^{2m}\widetilde{I}_{2m+2\nu}(w)C_m^{\nu}(t), \qquad \mathrm{Re}(\nu)>0 \tag{4.41}$$

or equivalently,

$$w^{2\nu}\widetilde{J}_{\nu-1/2}\Big(\frac{w(1+t)^{1/2}}{\sqrt{2}}\Big) = \frac{2^{4\nu}\Gamma(\nu)}{\sqrt{\pi}}\sum_{m=0}^{\infty}(-1)^m(\nu+m)J_{2\nu+2m}(w)C_m^{\nu}(t).$$

The first formula (4.40) *is Gegenbauer's expansion (see for instance* [60, 7.13(14)]*), whereas the second expansion formula* (4.41) *was proved in Kobayashi–Mano* [36, Proposition 5.7.1].

4.3. **Integral representation for $\Omega_{k,a}(\gamma_z)$.**

In this section, we find the integral kernel $\Lambda_{k,a}(x,y;z)$ of the operator $\Omega_{k,a}(\gamma_z)$ for $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$. The main result is Theorem 4.23.

4.3.1. *The function $\mathscr{I}(b,\nu;w;t)$.*

In this subsection, we introduce a function $\mathscr{I}(b,\nu;w;t)$ of four variables, and study its basic properties.

Let $\widetilde{I}_\lambda(w) = (\frac{w}{2})^{-\lambda}I_\lambda(w)$ be the normalized I-Bessel function (see (4.9)), and $C_m^{\nu}(t)$ the Gegenbauer polynomial. Consider the following infinite sum:

$$\mathscr{I}(b,\nu;w;t) = \frac{\Gamma(b\nu+1)}{\nu}\sum_{m=0}^{\infty}(m+\nu)\left(\frac{w}{2}\right)^{bm}\widetilde{I}_{b(m+\nu)}(w)C_m^{\nu}(t). \tag{4.42}$$

We note that $\nu=0$ is not a singularity in the summand because $C_m^0(t)\equiv 0$ for $m\ge 1$ (see Fact 4.8 2); see also (4.31)). In this subsection, we prove:

Lemma 4.17. 1) *The summation* (4.42) *converges absolutely and uniformly on any compact subset of*

$$U := \{(b,\nu,w,t)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{C}\times[-1,1] : 1+b\nu>0\}. \tag{4.43}$$

In particular, $\mathscr{I}(b,\nu;w;t)$ is a continuous function on U.

2) *(Special value at $w=0$)*

$$\mathscr{I}(b,\nu;0;t)\equiv 1. \tag{4.44}$$

3) *(Gegenbauer transform) For $\nu > -\frac{1}{2}$,*

$$\mathscr{C}_{\nu,m}(\mathscr{I}(b,\nu;w;\cdot)) = \Gamma(1+b\nu)\left(\frac{w}{2}\right)^{bm}\widetilde{I}_{b(m+\nu)}(w), \quad \textit{for } m\in\mathbb{N}.$$

Proof. 1) It is sufficient to show that for a sufficiently large m_0 the summation over m $(\ge m_0)$ converges absolutely and uniformly on any compact set of U. We recall from (4.16) and

(4.29) that

$$|\widetilde{I}_\lambda(w)| \le \frac{e^{|\operatorname{Re} w|}}{\Gamma(\lambda+1)},$$
$$\left|\frac{1}{\nu}C_m^\nu(t)\right| \le B(\nu)m^{2\nu-1} \quad \text{for any } m \ge 1.$$

Then,

$$\begin{aligned}\frac{1}{\nu}\sum_{m=m_0}^{\infty}\left|(m+\nu)\left(\frac{w}{2}\right)^{bm}\widetilde{I}_{b(m+\nu)}(w)C_m^\nu(t)\right| &\le \sum_{m=m_0}^{\infty}\left|\frac{(m+\nu)B(\nu)w^{bm}e^{|\operatorname{Re} w|}m^{2\nu-1}}{2^{bm}\Gamma(bm+b\nu+1)}\right| \\ &= B(\nu)e^{|\operatorname{Re} w|}\sum_{m=m_0}^{\infty}\frac{(1+\frac{\nu}{m})m^{2\nu}}{\Gamma(bm+b\nu+1)}\left(\left|\frac{w}{2}\right|^b\right)^m.\end{aligned}$$

Since $b > 0$, $\Gamma(bm + b\nu + 1)$ grows faster than any other term in each summand as m goes to infinity, and consequently, the last sum converges. Furthermore, the convergence is uniform on any compact set of parameters (b, ν, w). Hence, we have proved the first assertion.

2) Since $b > 0$, the summand in (4.42) vanishes at $w = 0$ for any $m > 0$, and therefore

$$\begin{aligned}\mathscr{I}(b,\nu;0;t) &= \frac{\Gamma(b\nu+1)}{\nu}\cdot\nu\cdot\widetilde{I}_{b\nu}(0)\cdot C_0^\nu(t) \\ &= 1.\end{aligned}$$

Thus, the second assertion is proved.

3) This is an immediate consequence of Fact 4.12 on the Gegenbauer transform $\mathscr{C}_{\nu,m}$. ◻

Example 4.18. *The special values at* $b = 1, 2$ *are given by*

$$\mathscr{I}(1,\nu;w;t) = e^{wt}, \tag{4.45}$$

$$\mathscr{I}(2,\nu;w;t) = \Gamma\Big(\nu+\frac{1}{2}\Big)\widetilde{I}_{\nu-\frac{1}{2}}\left(\frac{w(1+t)^{\frac{1}{2}}}{\sqrt{2}}\right). \tag{4.46}$$

Proof of Example 4.18. First, let us prove the identity (4.45). By Lemma 4.17 3), we have

$$\mathscr{C}_{\nu,m}(\mathscr{I}(b,\nu;w;\cdot)) = \Gamma(1+b\nu)\left(\frac{w}{2}\right)^{bm}\widetilde{I}_{b(m+\nu)}(w), \quad \text{for all } m \in \mathbb{N}.$$

By Lemma 4.14 2), we have

$$\mathscr{C}_{\nu,m}(e^{wt}) = \Gamma(1+\nu)\left(\frac{w}{2}\right)^m\widetilde{I}_{\nu+m}(w).$$

This shows that

$$\mathscr{C}_{\nu,m}(\text{left-hand side}) = \mathscr{C}_{\nu,m}(\text{right-hand side}) \quad \text{for all } m \in \mathbb{N}$$

with regard to the identity (4.45). Since the Gegenbauer polynomials form a complete orthogonal basis in the Hilbert space $L^2((-1,1),(1-t^2)^{\nu-\frac{1}{2}}dt)$ (see Fact 4.12), we have proved (4.45).

The proof for the identity (4.46) goes similarly by using Lemma 4.14 1). Thus, Example 4.18 has been shown. ◻

4.3.2. *The normalization constant.*
For $a > 0$ and a multiplicity function k on the root system $\mathscr{R}$, we introduce the following normalization constant

$$c_{k,a} := \Big(\int_{\mathbb{R}^N} \exp\Big(-\frac{1}{a}\|x\|^a\Big)\vartheta_{k,a}(x)dx\Big)^{-1} \tag{4.47}$$

where $\vartheta_{k,a}$ is the density defined in (1.2). Using the polar coordinates, we have

$$\begin{aligned} c_{k,a}^{-1} &= \int_0^\infty \int_{S^{N-1}} \exp\Big(-\frac{1}{a}r^a\Big) r^{2\langle k\rangle+N+a-3}\vartheta_k(\omega)d\sigma(\omega)dr \\ &= d_k^{-1}\int_0^\infty e^{-t}(at)^{\frac{N+2\langle k\rangle-2}{a}}dt. \end{aligned}$$

Here, d_k^{-1} is the k-deformation of the volume of the unit sphere (see (2.12)). For a non-negative multiplicity function k, the integral converges if $2\langle k\rangle + N + a - 2 > 0$, and we get

$$c_{k,a} = a^{-(\frac{2\langle k\rangle+N-2}{a})}\Gamma(\frac{2\langle k\rangle + N + a - 2}{a})^{-1}d_k. \tag{4.48}$$

For $k \equiv 0$, we have $d_0^{-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ (see (2.13)), and in particular,

$$c_{0,1} = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}\Gamma(N-1)} = \frac{1}{(4\pi)^{\frac{N-1}{2}}\Gamma(\frac{N-1}{2})}, \quad c_{0,2} = \frac{1}{(2\pi)^{\frac{N}{2}}}.$$

4.3.3. *Definition of* $h_{k,a}(r,s;z;t)$ *and* $\Lambda_{k,a}(x,y;z)$.
We now introduce the following continuous function of t on the interval $[-1,1]$ with parameters $r, s > 0$, and $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$:

$$h_{k,a}(r,s;z;t) := \frac{\exp(-\frac{1}{a}(r^a+s^a)\coth(z))}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}} \mathscr{I}\left(\frac{2}{a}, \frac{2\langle k\rangle+N-2}{2}; \frac{2(rs)^{\frac{a}{2}}}{a\sinh(z)}; t\right). \tag{4.49}$$

We observe that, for $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$, the substitution $z = i\mu$ into (4.49) yields:

$$h_{k,a}(r,s;i\mu;t) = \frac{\exp(\frac{i}{a}(r^a+s^a)\cot(\mu))}{e^{\frac{2\langle k\rangle+N+a-2}{2a}\pi i}\sin(\mu)^{\frac{2\langle k\rangle+N+a-2}{a}}} \mathscr{I}\left(\frac{2}{a}, \frac{2\langle k\rangle+N-2}{2}; \frac{2(rs)^{\frac{a}{2}}}{ai\sin(\mu)}; t\right). \tag{4.50}$$

We recall from (4.3) that $\Lambda_{k,a}^{(m)}(r,s;z)$ is the integral kernel of the operator $\Omega_{k,a}^{(m)}(\gamma_z)$ on $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$. Up to a constant factor (independent of m), the Gegenbauer transform of $h_{k,a}$ coincides with $\Lambda_{k,a}^{(m)}(r,s;z)$:

Lemma 4.19. *Suppose* $2\langle k\rangle + N > 1$. *Then, for every* $m \in \mathbb{N}$, *we have*

$$\begin{aligned} \mathscr{C}_{\langle k\rangle+\frac{N}{2}-1,m}(h_{k,a}(r,s;z;\cdot)) &= \frac{d_k}{c_{k,a}}\Lambda_{k,a}^{(m)}(r,s;z) \\ &= a^{\frac{2\langle k\rangle+N-2}{a}}\Gamma\left(\frac{2\langle k\rangle+N+a-2}{a}\right)\Lambda_{k,a}^{(m)}(r,s;z). \end{aligned} \tag{4.51}$$

Proof. We observe

$$1 + b\nu = \frac{2\langle k\rangle + a + N - 2}{a} \quad \text{and} \quad \lambda_{k,a,m} = \frac{2}{a}(m+\nu)$$

if $b = \frac{2}{a}$ and $\nu = \frac{2\langle k\rangle + N - 2}{2}$. Then, Lemma 4.19 follows from Lemma 4.17 3) and the definition (4.11) of $\Lambda^{(m)}_{k,a}(r, s; z)$. ◻

We are ready to define the following function on $\mathbb{R}^N \times \mathbb{R}^N \times (\mathbb{C}^+ \setminus i\pi\mathbb{Z})$ by

$$\Lambda_{k,a}(r\omega, s\eta; z) := (\widetilde{V}_k h_{k,a}(r, s; z; \cdot))(\omega, \eta), \tag{4.52}$$

where $\widetilde{V}_k$ is introduced in (2.6) by using the Dunkl intertwining operator V_k and $h_{k,a}(r, s; z; t)$ is defined in (4.49).

4.3.4. *Expansion formula.*
For $a > 0$, we will derive a series representation for the kernel $\Lambda_{k,a}$ in terms of $\Lambda^{(m)}_{k,a}$ and the Poisson kernel of the space $\mathscr{H}^m_k(\mathbb{R}^N)$.

In light of the definitions of $\mathscr{I}(b, \nu; w; t)$ (see (4.42)) and $\Lambda^{(m)}_{k,a}(r, s; z)$ (see (4.11)), we may rewrite (4.49) as

$$h_{k,a}(r, s; z; t) = a^{(\frac{2\langle k\rangle + N - 2}{a})}\Gamma\left(\frac{2\langle k\rangle + N + a - 2}{a}\right)\sum_{m\in\mathbb{N}} \Lambda^{(m)}_{k,a}(r, s; z)\Big(\frac{\langle k\rangle + m + \frac{N-2}{2}}{\langle k\rangle + \frac{N-2}{2}}\Big)C_m^{\langle k\rangle + \frac{N-2}{2}}(t). \tag{4.53}$$

The above expansion formula (4.53) is the series expansion by Gegenbauer polynomials (Fact 4.12) corresponding to Lemma 4.19.

Now, applying the operator $\widetilde{V}_k$ to (4.53), we get

Theorem 4.20. *For $a > 0$ and $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$, we have*

$$\Lambda_{k,a}(x, y; z) = a^{(\frac{2\langle k\rangle + N - 2}{a})}\Gamma\left(\frac{2\langle k\rangle + N + a - 2}{a}\right)\sum_{m\in\mathbb{N}} \Lambda^{(m)}_{k,a}(r, s; z)P_{k,m}(\omega, \eta)$$

where $x = r\omega$, $y = s\eta$, and

$$P_{k,m}(\omega, \eta) := \Big(\frac{\langle k\rangle + m + \frac{N-2}{2}}{\langle k\rangle + \frac{N-2}{2}}\Big)\Big(\widetilde{V}_k C_m^{\langle k\rangle + \frac{N-2}{2}}\Big)(\omega, \eta). \tag{4.54}$$

In Theorem 4.20, the function $P_{k,m}(\omega, \eta)$ on $S^{N-1} \times S^{N-1}$ is the *Poisson kernel*, or the reproducing kernel, of the space of spherical k-harmonic polynomials of degree m, which is characterized by the following proposition.

Proposition 4.21. *$P_{k,m}(\omega, \eta)$ is the kernel function of the projection from the Hilbert space $L^2(S^{N-1}, \vartheta_k(\eta)d\eta)$ to $\mathscr{H}^m_k(\mathbb{R}^N)$, namely, for any $p \in \mathscr{H}^n_k(\mathbb{R}^N)$,*

$$d_k \int_{S^{N-1}} P_{k,m}(\omega, \eta)p(\eta)\vartheta_k(\eta)d\sigma(\eta) = \begin{cases} p(\omega) & (n = m), \\ 0 & (n \neq m). \end{cases}$$

Example 4.22. *For $N = 1$, S^{N-1} consists of two points (± 1), and $\mathscr{H}_k^m(\mathbb{R}^N) = 0$ if $m \geq 2$. In this case, it is easy to see*

$$d_k = \frac{1}{2},$$
$$P_{k,m}(\omega, \eta) = \begin{cases} 1 & (m = 0), \\ \operatorname{sgn}(\omega\eta) & (m = 1). \end{cases}$$

Proof of Proposition 4.21. By the Funk–Hecke formula in the Dunkl setting (see Fact 4.13), we have

$$d_k \int_{S^{N-1}} \left(\widetilde{V}_k C_m^{\langle k\rangle + \frac{N-2}{2}}\right)(\omega, \eta) p(\eta) \vartheta_k(\eta) d\sigma(\eta) = \mathscr{C}_{\langle k\rangle + \frac{N-2}{2}, n} \left(C_m^{\langle k\rangle + \frac{N-2}{2}}\right) p(\omega)$$
$$= \begin{cases} \frac{\langle k\rangle + \frac{N-2}{2}}{m + \langle k\rangle + \frac{N-2}{2}} p(\omega) & (n = m), \\ 0 & (n \neq m). \end{cases}$$

Here, we have used Fact 4.12 and (4.35) for the last equality. Hence, Proposition 4.21 follows. $\square$

4.3.5. *Integral representation of $\Omega_{k,a}(\gamma_z)$.*
We are ready to prove the main result of this section. Recall from Theorem 3.39 that $\Omega_{k,a}(\gamma_z)$ is a holomorphic semigroup consisting of Hilbert–Schmidt operators on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ for $\operatorname{Re} z > 0$, and is a one-parameter subgroup of unitary operators for $z \in i\mathbb{R}$. Here is an integral representation of $\Omega_{k,a}(\gamma_z)$:

Theorem 4.23. *Suppose $a > 0$ and k is a non-negative multiplicity function on the root system $\mathscr{R}$ satisfying*

$$2\langle k\rangle + N > \max(1, 2 - a). \tag{4.55}$$

1) *Suppose $\operatorname{Re} z > 0$. Then, the Hilbert–Schmidt operator $\Omega_{k,a}(\gamma_z)$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is given by*

$$\Omega_{k,a}(\gamma_z) f(x) = c_{k,a} \int_{\mathbb{R}^N} f(y) \Lambda_{k,a}(x, y; z) \vartheta_{k,a}(y) dy, \tag{4.56}$$

where $c_{k,a}$ is the constant defined in (4.47) and the kernel function $\Lambda_{k,a}(x, y; z)$ is defined in (4.52).

2) *Suppose $z = i\mu$ ($\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$). Then, the unitary operator $\Omega_{k,a}(\gamma_{i\mu})$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is given by*

$$\Omega_{k,a}(\gamma_{i\mu}) f(x) = c_{k,a} \int_{\mathbb{R}^N} f(y) \Lambda_{k,a}(x, y; i\mu) \vartheta_{k,a}(y) dy. \tag{4.57}$$

Proof. Thanks to Proposition 4.1, it suffices to show the following identity:

$$c_{k,a} \int_{S^{N-1}} \Lambda_{k,a}(r\omega, s\eta; z) p(\eta) \vartheta_k(\eta) d\sigma(\eta) = \Lambda_{k,a}^{(m)}(r, s; z) p(\omega),$$

for all $p \in \mathscr{H}_k^m(\mathbb{R}^N)$ and $m \in \mathbb{N}$. This follows from Theorem 4.20, Proposition 4.21, and (4.48). Hence, Theorem 4.23 is proved. $\square$

4.4. **The $a = 1, 2$ case.**
As we have seen in Theorem 4.23, the kernel function $\Lambda_{k,a}(x, y; z)$ for the holomorphic semigroup $\Omega_{k,a}(\gamma_z)$ is given as

$$\Lambda_{k,a}(r\omega, s\eta; z) = (\widetilde{V}_k h_{k,a}(r, s; z; \cdot))(\omega, \eta).$$

See (2.6) for the definition of $\widetilde{V}_k$. In this section, we give a closed formula of $h_{k,a}(r, s; z; t)$ for $a = 1, 2$, and discuss the convergence of the integral (4.56) in Theorem 4.23.

4.4.1. *Explicit formula for $h_{k,a}(r, s; z; t)$ ($a = 1, 2$).*
When $a = 1, 2$, the series expansion in (4.53) can be expressed in terms of elementary functions as follow.

Theorem 4.24. *Let $\langle k\rangle$ be defined in (2.3), and $\widetilde{I}_\nu$ the normalized I-Bessel function (see (4.9)). Then, for $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$, we have:*

$$h_{k,a}(r, s; z; t) = \frac{\exp(-\frac{1}{a}(r^a + s^a)\coth(z))}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}} \times \begin{cases} \Gamma\Big(\langle k\rangle + \dfrac{N-1}{2}\Big)\widetilde{I}_{\langle k\rangle+\frac{N-3}{2}}\Big(\dfrac{\sqrt{2}(rs)^{\frac{1}{2}}}{\sinh z}(1+t)^{\frac{1}{2}}\Big) & (a = 1), \\ \exp\Big(\dfrac{rst}{\sinh z}\Big) & (a = 2). \end{cases} \tag{4.58}$$

Proof. In view of the definition (4.49) of $h_{k,a}(r, s; z; t)$, Theorem 4.24 follows from formulas (4.45) and (4.46) for $\mathscr{I}(a, \nu; w; t)$ in Example 4.18. □

4.4.2. *Absolute convergence of integral representation.*
By using Theorem 4.24, we shall give an upper bound for $\Lambda_{k,a}(x, y; z)$. We begin with the following:

Lemma 4.25. *For $b = 1, 2$*

$$|\mathscr{I}(b, \nu; w; t)| \le e^{|\operatorname{Re} w|} \tag{4.59}$$

for any $t \in [-1, 1]$, $\nu > 0$ and $w \in \mathbb{C}$.

Proof. We have seen in (4.45) and (4.46) the explicit formulas of $\mathscr{I}(b, \nu; w; t)$ for $b = 1, 2$. Then (4.59) is obvious for $b = 1$, and follows from the upper estimate (4.16) of the I-Bessel function for $b = 2$. □

Proposition 4.26. *Suppose b is a positive number, for which the inequality (4.59) holds. Let $a := \frac{2}{b}$. Then the function $\Lambda_{k,a}(x, y; z)$ (see (4.52)) satisfies the following inequalities:*
1) *For $\operatorname{Re} z > 0$, there exist positive constants A, B depending on z such that*

$$|\Lambda_{k,a}(x, y; z)| \le A \exp\left(-B(\|x\|^a + \|y\|^a)\right), \quad \text{for any } x, y \in \mathbb{R}^N. \tag{4.60}$$

2) *For $z = i\mu + \epsilon$ ($\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$, $\epsilon \ge 0$),*

$$|\Lambda_{k,a}(x, y; i\mu + \epsilon)| \le \frac{1}{|\sin(\mu)|^{\frac{N+2\langle k\rangle+a-2}{a}}}. \tag{4.61}$$

Remark 4.27. *By Lemma 4.25, the assumption of Proposition 4.26 is fulfilled for $b = 1, 2$. We do not know if (4.59) holds for b other than 1 and 2.*

Proof. Suppose the inequality (4.59) of $\mathscr{I}(b, \nu; w; t)$ holds. Then, by the definition of $h_{k,a}(r, s; z; t)$ in (4.49), the inequality (4.59) brings us to the following estimate:

$$|h_{k,a}(r, s; z; t)| \leq \frac{1}{|\sinh(z)|^{\frac{2\langle k\rangle+N+a-2}{a}}} \exp\Bigl(-\frac{1}{a}(r^a + s^a)\alpha(z)\Bigr) \exp\Bigl(\frac{2}{a}(rs)^{\frac{a}{2}}\alpha(z)\beta(z)\Bigr).$$

Here, we have used the functions $\alpha(z)$ and $\beta(z)$ defined in Lemma 4.2. Since $r^a + s^a \geq 2(rs)^{\frac{a}{2}}$, we have obtained:

$$|h_{k,a}(r, s; z; t)| \leq \frac{1}{|\sinh(z)|^{\frac{N+2\langle k\rangle+a-2}{a}}} \exp\Bigl(-\frac{1}{a}(r^a + s^a)\alpha(z)(1 - |\beta(z)|)\Bigr).$$

We recall that $\Lambda_{k,a}(x, y; z)$ is defined by applying the operator $\widetilde{V}_k$ to $h_{k,a}(r, s; z; \cdot)$ (see (4.52)). Then, it follows from Proposition 2.5 that

$$|\Lambda_{k,a}(x, y; z)| \leq \|h_{k,a}(r, s; z; \cdot)\|_{L^\infty} \leq \frac{1}{|\sinh(z)|^{\frac{N+2\langle k\rangle+a-2}{a}}} \exp\Bigl(-\frac{1}{a}(\|x\|^a + \|y\|^a)\alpha(z)(1 - |\beta(z)|)\Bigr).$$

Suppose now that $\operatorname{Re} z > 0$. Then, $\alpha(z) > 0$ and $|\beta(z)| < 1$ by Lemma 4.2. Hence, we have proved (4.60).

On the other hand, suppose $z = i\mu + \epsilon$ ($\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$, $\epsilon \geq 0$). Then $\alpha(z) \geq 0$, $|\beta(z)| < 1$ by Lemma 4.2, and as we have seen in (4.23)

$$|\sinh z| \geq |\sin \mu|.$$

Hence, we have shown (4.61). □

Now we are ready to prove:

Corollary 4.28. *Suppose we are in the setting of Theorem 4.23. Let $a = 1, 2$.*

1) *For* $\operatorname{Re} z > 0$, *the right-hand side of* (4.56) *converges absolutely for any* $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.
2) *For* $z = i\mu \in i(\mathbb{R} \setminus \pi\mathbb{Z})$, *the right-hand side of* (4.57) *converges absolutely for all* $f \in (L^1 \cap L^2)(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

Proof. 1) It follows from Proposition 4.26 1) that

$$\Lambda_{k,a}(x, y; z) \in L^2(\mathbb{R}^N \times \mathbb{R}^N, \vartheta_{k,a}(x)\vartheta_{k,a}(y)dx\,dy)$$

for $\operatorname{Re} z > 0$. Therefore, Corollary is clear from the Cauchy–Schwartz inequality.

2) We shall substitute z by $\epsilon + i\mu$ in (4.56) and let ϵ goes to 0.

For the left-hand side of (4.56), we use Theorem 3.39 1), and get

$$\lim_{\epsilon\to 0} \Omega_{k,a}(\gamma_{\epsilon+i\mu}) = \Omega_{k,a}(\gamma_{i\mu}).$$

For the right-hand side of (4.56), thanks to Proposition 4.26 2), we see

$$\lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^N} \Lambda_{k,a}(x, y; i\mu + \epsilon) f(y)\vartheta_{k,a}(y)dy = \int_{\mathbb{R}^N} \Lambda_{k,a}(x, y; i\mu) f(y)\vartheta_{k,a}(y)dy$$

for $f \in L^1(\mathbb{R}^N, \vartheta_{k,a}(y)dy)$ by the Lebesgue dominated convergence theorem. Hence, we have shown that

$$(\Omega_{k,a}(\gamma_{i\mu}) f)(x) = \int_{\mathbb{R}^N} \Lambda_{k,a}(x, y; i\mu) f(y)\vartheta_{k,a}(y)dy,$$

and the right-hand side converges for any $f \in (L^1 \cap L^2)(\mathbb{R}^N, \vartheta_{k,a}(y)dy)$. Hence, Corollary 4.28 has been proved. □

4.5. **The rank one case.**

For the one dimensional case, the only choice of the non-trivial reduced root system $\mathscr{R}$ is $\mathscr{R} = \{\pm 1\}$ in $\mathbb{R}$ up to scaling, corresponding to the Coxeter group $\mathfrak{C} = \{\mathrm{id}, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$, where $\sigma(x) = -x$. Here $\langle k \rangle = k$. In this section we give a closed form of $\Lambda_{k,a}(x, y; z)$ for $N = 1$.

First of all, for $N = 1$ and $a > 0$, we note that we do not need Lemma 4.19, for which the assumption was $2\langle k \rangle + N > 1$. Hence, instead of (4.55), we simply need the following assumption:

$$a > 0 \quad \text{and} \quad 2k > 1 - a. \tag{4.62}$$

The goal of this section is to find a closed formula of the kernel function $\Lambda_{k,a}(x, y; z)$ (see (4.52)) for all $a > 0$ and for an arbitrary multiplicity function subject to (4.62).

Proposition 4.29. *Let $N = 1$, $a > 0$, $k \geq 0$ and $2k > 1 - a$. For $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$, the holomorphic semigroup $\Omega_{k,a}(\gamma_z)$ on $L^2(\mathbb{R}, |x|^{2k+a-2}dx)$ is given by*

$$\Omega_{k,a}(\gamma_z) f(x) = 2^{-1} a^{-(\frac{2k-1}{a})} \Gamma\Big(\frac{2k + a - 1}{a}\Big)^{-1} \int_{\mathbb{R}} f(y) \Lambda_{k,a}(x, y; z) |y|^{2k+a-2} dy,$$

where

$$\Lambda_{k,a}(x, y; z) = \Gamma\Big(\frac{2k + a - 1}{a}\Big) \frac{e^{-\frac{1}{a}(|x|^a + |y|^a)\coth(z)}}{\sinh(z)^{\frac{2k+a-1}{a}}} \Big(\widetilde{I}_{\frac{2k-1}{a}}\Big(\frac{2}{a} \frac{|xy|^{\frac{a}{2}}}{\sinh(z)}\Big) + \frac{1}{a^{\frac{2}{a}}} \frac{xy}{\sinh(z)^{\frac{2}{a}}} \widetilde{I}_{\frac{2k+1}{a}}\Big(\frac{2}{a} \frac{|xy|^{\frac{a}{2}}}{\sinh(z)}\Big)\Big). \tag{4.63}$$

Here $\widetilde{I}_\nu(w)$ denotes the normalized Bessel function (4.9).

Proof. By Theorem 4.20, the kernel $\Lambda_{k,a}(x, y; z)$ can be recovered from a family of functions $\{\Lambda_{k,a}^{(m)}(r, s; z) : m \in \mathbb{N}\}$. In the rank one case, S^0 consists of two points, and correspondingly, Theorem 4.20 collapses to the following:

$$\begin{aligned}
&\Lambda_{k,a}(x, y; z) \\
= \;& \frac{c_{k,a}^{-1}}{2} \Big(\Lambda_{k,a}^{(0)}(|x|, |y|; z) + \Lambda_{k,a}^{(1)}(|x|, |y|; z) \operatorname{sgn}(xy)\Big) \\
= \;& \frac{c_{k,a}^{-1}}{2} \frac{e^{-\frac{1}{a}(|x|^a + |y|^a)\coth(z)}}{\sinh(z)} |xy|^{-k+(1/2)} \Big(I_{\lambda_{k,a,0}}\Big(\frac{2}{a} \frac{|xy|^{\frac{a}{2}}}{\sinh(z)}\Big) + I_{\lambda_{k,a,1}}\Big(\frac{2}{a} \frac{|xy|^{\frac{a}{2}}}{\sinh(z)}\Big) \operatorname{sgn}(xy)\Big) \\
= \;& \frac{c_{k,a}^{-1}}{2} \frac{e^{-\frac{1}{a}(|x|^a + |y|^a)\coth(z)}}{\sinh(z)} \Big(\frac{1}{(a \sinh(z))^{\lambda_{k,a,0}}} \widetilde{I}_{\lambda_{k,a,0}}\Big(\frac{2}{a} \frac{|xy|^{\frac{a}{2}}}{\sinh(z)}\Big) + \frac{xy}{(a \sinh(z))^{\lambda_{k,a,1}}} \widetilde{I}_{\lambda_{k,a,1}}\Big(\frac{2}{a} \frac{|xy|^{\frac{a}{2}}}{\sinh(z)}\Big)\Big),
\end{aligned}$$

where

$$\lambda_{k,a,0} = \frac{2k - 1}{a}, \quad \lambda_{k,a,1} = \frac{2k + 1}{a}, \quad c_{k,a}^{-1} = 2a^{(\frac{2k-1}{a})} \Gamma\Big(\frac{2k + a - 1}{a}\Big).$$

Here, we have used Example 4.22 for the first equality, and the formula (4.20) of $\Lambda_{k,a}^{(m)}$ for the second equality. This finishes the proof of Proposition 4.29. □

5. The (k,a)-generalized Fourier transforms $\mathscr{F}_{k,a}$

The object of our study in this section is the (k,a)-generalized Fourier transform given by

$$\mathscr{F}_{k,a} = e^{\frac{\pi i}{2}(\frac{2\langle k\rangle+N+a-2}{a})} \exp\Big(\frac{\pi i}{2a}(\|x\|^{2-a}\Delta_k - \|x\|^a)\Big).$$

This is a unitary operator on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

As we mentioned in Introduction, the unitary operator $\mathscr{F}_{k,a}$ includes some known transforms as special cases:

- the Euclidean Fourier transform [29] $(a = 2,\ k \equiv 0)$,
- the Hankel transform [36] $(a = 1,\ k \equiv 0)$,
- the Dunkl transform $\mathscr{D}_k$ $(a = 2,\ k > 0)$.

In this section, we develop the theory of the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ for general a and k by using the aforementioned idea of $\mathfrak{sl}_2$-triple. The point of our approach is that we interpret $\mathscr{F}_{k,a}$ not as an isolated operator but as a special value of the unitary representation $\Omega_{k,a}$ of the simply connected, simple Lie group $\widetilde{SL(2,\mathbb{R})}$ at $\gamma_{\frac{\pi i}{2}}$ (see (3.37)), or as the boundary value of the holomorphic semigroup. Then, we see that many properties of the Euclidean Fourier transforms can be extended to the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ by using the representation theory of $\widetilde{SL(2,\mathbb{R})}$. Our theorems for $\mathscr{F}_{k,a}$ include the inversion formula, and a generalization of the Plancherel formula, the Hecke formula, the Bochner formula, and the Heisenberg inequality for the uncertainty principle.

As in Diagram 1.4 of Introduction, the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ admits symmetries of $\mathfrak{C}\times\widetilde{SL(2,\mathbb{R})}$ for general (k,a), and even higher symmetries than $\mathfrak{C}\times\widetilde{SL(2,\mathbb{R})}$ for particular values of (k,a). In fact, if $k \equiv 0$, then the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is a representation space of the Schrödinger model of the Weil representation (see [18] and references therein) of the metaplectic group $Mp(N,\mathbb{R})$ for $a = 2$, and the L^2-model of the minimal representation (see [39]) of the conformal group $O(N+1,2)$ for $a = 1$. The special value $a = 2$ has a particular meaning also for general k in the sense that $\mathscr{F}_{k,2}$ is equal to the Dunkl transform $\mathscr{D}_k$. How about the $a = 1$ case for general k? The unitary operator

$$\mathscr{H}_k := \mathscr{F}_{k,1} \tag{5.1}$$

may be regarded as the Dunkl analogue of the Hankel-type transform $\mathscr{F}_{0,1}$ (see Diagram 1 in Introduction). As we have seen in Section 4.4, this unitary operator $\mathscr{H}_k$ can be written by means of the Dunkl intertwining operator V_k and the classical Bessel functions (see Section 5.3).

5.1. $\mathscr{F}_{k,a}$ as an inversion unitary element.

The (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is defined as

$$\mathscr{F}_{k,a} := e^{i\frac{\pi}{2}(\frac{2\langle k\rangle+N+a-2}{a})}\Omega_{k,a}(\gamma_{i\frac{\pi}{2}}). \tag{5.2}$$

Here, we recall from (3.37) that

$$\gamma_{\frac{\pi i}{2}} = \operatorname{Exp}\Big(\frac{\pi}{2i}\mathbf{k}\Big) = \operatorname{Exp}\Big(\frac{\pi}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Big)$$

is an element of the simply connected Lie group $\widetilde{SL(2,\mathbb{R})}$, and from Theorem 3.30 that $\Omega_{k,a}$ is a unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

In this subsection, we discuss basic properties of $\mathscr{F}_{k,a}$ for general k and a, which are derived from the fact that $\gamma_{\frac{\pi i}{2}}$ is a representative of the non-trivial (therefore, the longest) element of the Weyl group for $\mathfrak{sl}_2$.

Theorem 5.1. *Let $a > 0$ and k be a non-negative multiplicity function on the root system $\mathscr{R}$ satisfying the inequality $a + 2\langle k\rangle + N > 2$ (see* (3.30)*).*

(1) (Plancherel formula) *The (k, a)-generalized Fourier transform $\mathscr{F}_{k,a} : L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \to L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is a unitary operator. That is, $\mathscr{F}_{k,a}$ is a bijective linear operator satisfying*
$$\|\mathscr{F}_{k,a}(f)\|_k = \|f\|_k \quad \textit{for any } f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx).$$
(2) *We recall from* (3.28) *that $\Phi^{(a)}_\ell(p,\cdot)$ is a function on $\mathbb{R}^N$ defined as*
$$\Phi^{(a)}_\ell(p, x) = p(x)L^{(\lambda_{k,a,m})}_\ell\Big(\frac{2}{a}\|x\|^a\Big)\exp\Big(-\frac{1}{a}\|x\|^a\Big), \quad x \in \mathbb{R}^N,$$
for $\ell, m \in \mathbb{N}$ and $p \in \mathscr{H}^m_k(\mathbb{R}^N)$. Then, $\Phi^{(a)}_\ell(p,\cdot)$ is an eigenfunction of $\mathscr{F}_{k,a}$:
$$\mathscr{F}_{k,a}(\Phi^{(a)}_\ell(p,\cdot)) = e^{-i\pi(\ell+\frac{m}{a})}\Phi^{(a)}_\ell(p,\cdot). \tag{5.3}$$

Proof. Since the phase factor in (5.2) is modulus one, the first statement is an immediate consequence of the fact that $\Omega_{k,a}$ is a unitary representation of $S\widetilde{L(2,\mathbb{R})}$ (see Theorem 3.30).

To see the second statement, we recall from Proposition 3.12 1) and Theorem 3.19 that $\Phi^{(a)}_\ell(p,\cdot)$ is an eigenfunction of $\omega_{k,a}(\mathbf{k})$. Then, the integration of (3.33 a) leads us to the identity (5.3). □

Corollary 5.2. *The (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$. If a is of the form $a = \frac{q}{q'}$, where q and q' are positive integers, then*
$$(\mathscr{F}_{k,a})^{2q} = \mathrm{id}\,.$$

Proof. We recall from Proposition 3.12 3) that
$$W_{k,a}(\mathbb{R}^N) = \mathbb{C}\text{-span}\{\Phi^{(a)}_\ell(p,\cdot) \mid \ell \in \mathbb{N}, m \in \mathbb{N}, p \in \mathscr{H}^m_k(\mathbb{R}^N)\}$$
is a dense subspace in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. Hence, it follows from (5.3) that the unitary operator $\mathscr{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$. If $a = \frac{q}{q'}$, then $(\mathscr{F}_{k,a})^{2q}$ acts on $\Phi^{(a)}_\ell(p,\cdot)$ as a scalar multiplication by
$$\Big(e^{-i\pi(\ell+\frac{m}{a})}\Big)^{2q} = 1$$
for any $m \in \mathbb{N}$ and any $p \in \mathscr{H}^m_k(\mathbb{R}^N)$. Thus, we have proved $(\mathscr{F}_{k,a})^{2q} = \mathrm{id}$. □

Corollary 5.2 implies particularly that $\mathscr{H}_k := \mathscr{F}_{k,1}$ (see (5.1)) is of order two, and the Dunkl transform $\mathscr{D}_k = \mathscr{F}_{k,2}$ is of order four. We pin down these particular cases as follows:

Theorem 5.3 (inversion formula)**.** *Let k be a non-negative multiplicity function on the root system $\mathscr{R}$.*

1) *Let r be any positive integer. Suppose $2\langle k\rangle + N > 2 - \frac{1}{r}$. Then, $\mathscr{F}_{k,\frac{1}{r}}$ is an involutive unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,\frac{1}{r}}(x)dx)$. Namely, the inversion formula is given by*
$$(\mathscr{F}_{k,\frac{1}{r}})^{-1} = \mathscr{F}_{k,\frac{1}{r}}. \tag{5.4}$$

2) *Let r be any non-negative integer. Suppose*

$$2\langle k\rangle + N > 2 - \frac{2}{2r+1}.$$

Then, $\mathscr{F}_{k,\frac{2}{2r+1}}$ is a unitary operator of order four on $L^2(\mathbb{R}^N, \vartheta_{k,\frac{2}{2r+1}}(x)dx)$. The inversion formula is given as

$$(\mathscr{F}^{-1}_{k,\frac{2}{2r+1}} f)(x) = (\mathscr{F}_{k,\frac{2}{2r+1}} f)(-x). \tag{5.5}$$

Proof. The first statement has been already proved. In the second statement, we remark that the inversion formula (5.5) is stronger than the fact that $(\mathscr{F}_{k,a})^4 = \mathrm{id}$ for $a = \frac{2}{2r+1}$. To see (5.5), we use (5.3) to get

$$\begin{aligned}(\mathscr{F}_{k,a})^2 \Phi^{(a)}_\ell(p,\cdot) &= \exp(-m(2r+1)\pi i)\Phi^{(a)}_\ell(p,\cdot)\\ &= (-1)^m \Phi^{(a)}_\ell(p,\cdot)\end{aligned}$$

if $a = \frac{2}{2r+1}$. Since $p(-x) = (-1)^m p(x)$ for $p \in \mathscr{H}^m_k(\mathbb{R}^N)$, we have shown that (5.5) holds for any $f \in W_{k,a}(\mathbb{R}^N)$. Since $W_{k,a}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$, we have proved (5.5). □

Remark 5.4. *Theorem 5.3 2) for $r = 0$ (i.e. $\mathscr{F}_{k,2} = \mathscr{D}_k$, the Dunkl transform) was proved in Dunkl* [10]*, and followed by de Jeu* [31] *where the author proved the inversion formula for $\mathbb{C}^+$-valued root multiplicity functions k. Our approach based on the SL_2 representation theory gives a new proof of the inversion formula and the Plancherel formula for $\mathscr{F}_{k,a}$ even for $a = 2$.*

Remark 5.5. *We recall from Theorem 3.31 that $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ decomposes into a discrete direct sum of irreducible unitary representations of $G = \widetilde{SL(2,\mathbb{R})}$. Hence, the square $(\mathscr{F}_{k,a})^2$ acts as a scalar multiple on each summand of* (3.43) *by Schur's lemma because $\gamma_{\pi i}$ is a central element of G (see* (3.38)*) and $(\mathscr{F}_{k,a})^2 = e^{i\pi(\frac{2\langle k\rangle+N+a-2}{a})}\Omega_{k,a}(\gamma_{\pi i})$ by* (5.2)*. Since $\gamma_{\pi i}$ acts on the irreducible representation $\pi(\lambda_{k,a,m})$ as a scalar $e^{-\pi i(\lambda_{k,a,m}+1)}$ by Fact 3.27 (5), $\mathscr{F}^2_{k,a}$ acts on it as the scalar*

$$e^{i\pi(\frac{2\langle k\rangle+N+a-2}{a})} e^{-\pi i(\lambda_{k,a,m}+1)} = e^{-\frac{2m\pi i}{a}}.$$

This gives us an alternative proof of Corollary 5.2.

Next, we discuss intertwining properties of the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ with differential operators. Let $E = \sum_{j=1}^N x_j\partial_j$ be the Euler operator on $\mathbb{R}^N$ as before.

Theorem 5.6. *The unitary operator $\mathscr{F}_{k,a}$ satisfies the following intertwining relations on a dense subspace of $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$:*

1) $\mathscr{F}_{k,a} \circ E = -(E + N + 2\langle k\rangle + a - 2) \circ \mathscr{F}_{k,a}$.
2) $\mathscr{F}_{k,a} \circ \|x\|^a = -\|x\|^{2-a}\Delta_k \circ \mathscr{F}_{k,a}$.
3) $\mathscr{F}_{k,a} \circ \|x\|^{2-a}\Delta_k = -\|x\|^a \circ \mathscr{F}_{k,a}$.

These identities hold in the usual sense, and also in the distribution sense in the space of distribution vectors of the unitary representation of G on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

If we use ξ (instead of x) for the coordinates of the target space of $\mathscr{F}_{k,a}$, we may write Theorem 5.6 2) and 3) as

$$\mathscr{F}_{k,a}(\|\cdot\|^a f)(\xi) = -\|\xi\|^{2-a}\Delta_k \mathscr{F}_{k,a}(f)(\xi), \tag{5.6 a}$$

$$\mathscr{F}_{k,a}(\|\cdot\|^{2-a}\Delta_k f)(\xi) = -\|\xi\|^a \mathscr{F}_{k,a}(f)(\xi). \tag{5.6 b}$$

Proof of Theorem 5.6. We observe that $\gamma_{\frac{\pi i}{2}}$ is a representative of the longest Weyl group element, and satisfies the following relations in $\mathfrak{sl}_2$:

$$\mathrm{Ad}(\gamma_{\frac{\pi i}{2}})\mathbf{h} = -\mathbf{h},\ \mathrm{Ad}(\gamma_{\frac{\pi i}{2}})\mathbf{e}^+ = -\mathbf{e}^-,\ \mathrm{Ad}(\gamma_{\frac{\pi i}{2}})\mathbf{e}^- = -\mathbf{e}^+,$$

(see (3.1) for the definition of $\mathbf{e}^+$, $\mathbf{e}^-$, and $\mathbf{h}$). In turn, we apply the identity

$$\Omega_{k,a}(g)\omega_{k,a}(X)\Omega_{k,a}(g)^{-1} = \omega_{k,a}(\mathrm{Ad}(g)X), \quad (g \in G,\ X \in \mathfrak{g}),$$

to $\mathbb{E}^+_{k,a} = \omega_{k,a}(\mathbf{e}^+)$, $\mathbb{E}^-_{k,a} = \omega_{k,a}(\mathbf{e}^-)$, and $\mathbb{H}_{k,a} = \omega_{k,a}(\mathbf{h})$ (see (3.6)). Then we have

$$\begin{aligned} \mathscr{F}_{k,a} \circ \mathbb{H}_{k,a} &= -\mathbb{H}_{k,a} \circ \mathscr{F}_{k,a}, \\ \mathscr{F}_{k,a} \circ \mathbb{E}^+_{k,a} &= -\mathbb{E}^-_{k,a} \circ \mathscr{F}_{k,a}, \\ \mathscr{F}_{k,a} \circ \mathbb{E}^-_{k,a} &= -\mathbb{E}^+_{k,a} \circ \mathscr{F}_{k,a}, \end{aligned} \tag{5.7}$$

because $\mathscr{F}_{k,a}$ is a constant multiple of $\Omega_{k,a}(\gamma_{\frac{\pi i}{2}})$ (see (5.2)). Now, Theorem 5.6 is read from the explicit formulas of $\mathbb{E}^+_{k,a}$, $\mathbb{E}^-_{k,a}$, and $\mathbb{H}_{k,a}$ (see (3.3)). □

5.2. **Density of** (k, a)**-generalized Fourier transform** $\mathscr{F}_{k,a}$**.**
By the Schwartz kernel theorem, the unitary operator $\mathscr{F}_{k,a}$ can be expressed by means of a distribution kernel. By using the normalizing constant $c_{k,a}$ (see (4.47)), we write the unitary operator $\mathscr{F}_{k,a}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ as an integral transform:

$$\mathscr{F}_{k,a} f(\xi) = c_{k,a} \int_{\mathbb{R}^N} B_{k,a}(\xi, x) f(x) \vartheta_{k,a}(x) dx. \tag{5.8}$$

Comparing this with the integral expression of $\Omega_{k,a}(\gamma_z)$ in Theorem 4.23, we see that the distribution kernel $B_{k,a}(\xi, x)$ in (5.8) is given by

$$B_{k,a}(x, y) = e^{i\pi(\frac{2\langle k\rangle+N+a-2}{2a})} \Lambda_{k,a}\Big(x, y; i\frac{\pi}{2}\Big) \tag{5.9}$$

because $\mathscr{F}_{k,a} = e^{i\pi(\frac{2\langle k\rangle+N+a-2}{2a})}\Omega_{k,a}(\gamma_{i\frac{\pi}{2}})$ (see (5.2)).

Now, Theorem 5.6 is reformulated as the differential equations that are satisfied by the distribution kernel $B_{k,a}(x, \xi)$ as follows:

Theorem 5.7. *The distribution* $B_{k,a}(\cdot, \cdot)$ *solves the following differential-difference equation on* $\mathbb{R}^N \times \mathbb{R}^N$

$$E^x B_{k,a}(\xi, x) = E^\xi B_{k,a}(\xi, x), \tag{5.10 a}$$

$$\|\xi\|^{2-a}\Delta_k^\xi B_{k,a}(\xi, x) = -\|x\|^a B_{k,a}(\xi, x), \tag{5.10 b}$$

$$\|x\|^{2-a}\Delta_k^x B_{k,a}(\xi, x) = -\|\xi\|^a B_{k,a}(\xi, x). \tag{5.10 c}$$

Here, the superscript in E^x, Δ_k^x, *etc indicates the relevant variable.*

Remark 5.8. *For* $a = 2$, *Theorem 5.7 was previously known as the differential equation of the Dunkl kernel (cf.* [9]*).*

Proof of Theorem 5.7. First we use the identity (5.7) as operators on $\mathbb{R}^N$ for any $a > 0$ and k. It is convenient to write

$$\mathbb{H}_{k,a} = \frac{1}{a}(\ell + 2E),$$

where E is the Euler operator and $\ell := N + 2\langle k\rangle + a - 2$. Then, by (5.8), the identity (5.7) implies

$$\int_{\mathbb{R}^N}\Big((\ell + 2E^x)f(x)\Big)B_{k,a}(\xi, x)\vartheta_{k,a}(x)dx = -\int_{\mathbb{R}^N} f(x)(\ell + 2E^\xi)B_{k,a}(\xi, x)\vartheta_{k,a}(x)dx \tag{5.11}$$

for any test function $f(x)$ (i.e. $f(x)\vartheta_{k,a}(x)^{\frac{1}{2}} \in \mathscr{S}(\mathbb{R}^N)$).

Now we recall that the density $\vartheta_{k,a}(x)$ (see (1.2) for definition) is homogeneous of degree $a - 2 + 2\langle k\rangle$ $(= \ell - N)$, we have

$$E^x\vartheta_{k,a}(x) = (\ell - N)\vartheta_{k,a}(x). \tag{5.12}$$

On the other hand, it follows from $\sum_{j=1}^N x_j \frac{\partial}{\partial x_j} - \sum_{j=1}^N \frac{\partial}{\partial x_j} x_j = -N$ as operators, we have

$$\int_{\mathbb{R}^N}(E^x f)(x)g(x)dx = -\int_{\mathbb{R}^N} f(x)(N + E^x)g(x)dx. \tag{5.13}$$

Combining (5.12) and (5.13), we have

$$\text{the left-hand side of (5.11)} = -\int_{\mathbb{R}^N} f(x)(\ell B_{k,a}(\xi, x) + 2E^x B_{k,a}(\xi, x))\vartheta_{k,a}(x)dx.$$

Hence, the identity (5.11) implies that the distribution kernel $B_{k,a}(\xi, x)$ satisfies the differential equation

$$E^x B_{k,a}(\xi, x) = E^\xi B_{k,a}(\xi, x). \tag{5.14}$$

Next, the identity (5.6 a) implies

$$\int_{\mathbb{R}^N} B_{k,a}(\xi, x)\|x\|^a f(x)\vartheta_{k,a}(x)dx = -\|\xi\|^{2-a}\Delta_k^\xi \int_{\mathbb{R}^N} B_{k,a}(\xi, x)f(x)\vartheta_{k,a}(x)dx$$

for any test function f. Hence the second differential equation (5.10 b) follows.

Finally, by the identity (5.6 b), we have

$$\int_{\mathbb{R}^N} B_{k,a}(\xi, x)\Big(\|x\|^{2-a}\Delta_k^x f(x)\Big)\vartheta_{k,a}(x)dx = -\|\xi\|^a \int_{\mathbb{R}^N} B_{k,a}(\xi, x)f(x)dx.$$

Since $\|x\|^{2-a}\Delta_k^x$ is a symmetric operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$, the left-hand side is equal to

$$\int_{\mathbb{R}^N} \|x\|^{2-a}\Big(\Delta_k^x B_{k,a}(\xi, x)\Big)f(x)\vartheta_{k,a}(x)dx.$$

Hence the third differential equation (5.10 c) is proved. $\square$

We continue basic properties on the kernel $B_{k,a}(\xi, x)$ of the (k, a) generalized Fourier transform.

Theorem 5.9.

1) $B_{k,a}(\lambda x, \xi) = B_{k,a}(x, \lambda\xi)$ *for* $\lambda > 0$.
2) $B_{k,a}(hx, h\xi) = B_{k,a}(x, \xi)$ *for* $h \in \mathfrak{C}$.
3) $B_{k,a}(\xi, x) = B_{k,a}(x, \xi)$.

4) $B_{k,a}(0, x) = 1$.

Proof. 1) This statement follows from the differential equation (5.10 a) given in Theorem 5.7.

2) Since $\mathscr{F}_{k,a}$ commutes with the action of the Coxeter group $\mathfrak{C}$, the second statement follows from the fact that $\vartheta_{k,a}(x)dx$ is a $\mathfrak{C}$-invariant measure.

3) Putting $\mu = \frac{\pi}{2}$ in (4.50), we get

$$h_{k,a}\Big(r, s; \frac{\pi i}{2}; t\Big) = e^{-\frac{2\langle k\rangle+N+a-2}{2a}\pi i}\mathscr{I}\left(\frac{2}{a}, \frac{2\langle k\rangle + N - 2}{2}; \frac{2(rs)^{\frac{a}{2}}}{ai}; t\right). \tag{5.15}$$

In particular, we have

$$h_{k,a}\Big(r, s; \frac{\pi i}{2}; t\Big) = h_{k,a}\Big(s, r; \frac{\pi i}{2}; t\Big).$$

In view of (4.52) and Proposition 2.5, we conclude that

$$\Lambda_{k,a}\Big(x, y; \frac{\pi i}{2}\Big) = \Lambda_{k,a}\Big(y, x; \frac{\pi i}{2}\Big).$$

Hence, the third statement has been proved.

4) By Lemma 4.17, $h_{k,a}(r, 0; \frac{\pi i}{2}; t) = e^{-\frac{2\langle k\rangle+N+a-2}{2a}\pi i}$. Since the Dunkl intertwining operator V_k satisfies $V_k(\mathbf{1}) = \mathbf{1}$ ($\mathbf{1}$ is the constant function on $\mathbb{R}^N$) (see (I2) in Section 2), it follows from (4.52) that $\Lambda_{k,a}(x, y; \frac{\pi i}{2}) = e^{-\frac{2\langle k\rangle+N+a-2}{2a}\pi i}$. Finally use (5.9). □

5.3. Generalized Fourier transform $\mathscr{F}_{k,a}$ for special values at $a = 1$ and 2.

In this subsection we discuss closed formulas of the kernel $B_{k,a}(x, y)$ of the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ (see (5.8)) in the case $a = 1, 2$. The (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ reduces to the Dunkl operator $\mathscr{D}_k$ if $a = 2$, and gives rise to a new unitary operator $\mathscr{H}_k$, the Dunkl analogue of the Hankel transform if $a = 1$.

We renormalize the Bessel function J_ν of the first kind as

$$\widetilde{J}_\nu(w) := \Big(\frac{w}{2}\Big)^{-\nu} J_\nu(w) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell w^{2\ell}}{2^{2\ell}\ell!\Gamma(\nu + \ell + 1)}. \tag{5.16}$$

Then, from the definition (4.9) of $\widetilde{I}_\nu(z)$ we have

$$\widetilde{J}_\nu(w) = \widetilde{I}_\nu(-iw) = \widetilde{I}_\nu(iw).$$

By substituting $z = \frac{\pi i}{2}$ into (4.58), we get the following formula:

$$h_{k,a}(r, s; \frac{\pi i}{2}; t) = \begin{cases} \Gamma\Big(\langle k\rangle + \frac{N-1}{2}\Big)e^{-\frac{\pi i}{2}(2\langle k\rangle+N-1)}\widetilde{J}_{\langle k\rangle+\frac{N-3}{2}}\big(\sqrt{2}(rs)^{\frac{1}{2}}(1 + t)^{\frac{1}{2}}\big) & (a = 1), \\ e^{-\frac{\pi i}{2}(\langle k\rangle+\frac{N}{2})}e^{-irst} & (a = 2). \end{cases}$$

Together with (5.9) and (4.52), we have:

Proposition 5.10. *In the polar coordinates $x = r\omega$ and $y = s\eta$, the kernel $B_{k,a}(x, y)$ is given by*

$$B_{k,a}(r\omega, s\eta) = \begin{cases} \Gamma\Big(\langle k\rangle + \frac{N-1}{2}\Big)\Big(\widetilde{V}_k\Big(\widetilde{J}_{\langle k\rangle+\frac{N-3}{2}}\big(\sqrt{2rs(1 + \cdot)}\big)\Big)\Big)(\omega, \eta) & (a = 1), \\ (\widetilde{V}_k(e^{-irs\cdot}))(\omega, \eta) & (a = 2). \end{cases} \tag{5.17}$$

As one can see form (5.17), the kernel $B_{k,2}(x, y)$ coincides with the Dunkl kernel at $(x, -iy)$ (cf. [11]).

Theorem 5.11. *Let k be a non-negative root multiplicity function, $a = 1$ or 2, and $x, y \in \mathbb{R}^N$. Then $|B_{k,a}(x,y)| \leq 1$.*

Proof. Theorem 5.11 follows from the special case, i.e. $\mu = \frac{\pi}{2}$, of Proposition 4.26 2) because $|B_{k,a}(x,y)| = |\Lambda_{k,a}(x,y; i\frac{\pi}{2})|$ by (5.9). □

Remark 5.12. *For $a = 2$ it was shown that $|B_{k,2}(x,y)|$ is uniformly bounded for $x, y \in \mathbb{R}^N$ first by de Jeu*[31] *and then by Rösler* [49] *by* 1.

We note that Theorem 5.11 implies the absolute convergence of the integral defining $\mathscr{F}_{k,a}$, for $a = 1, 2$, on $(L^1 \cap L^2)(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$, as we proved in Corollary 4.28.

5.4. Generalized Fourier transform $\mathscr{F}_{k,a}$ in the rank-one case.

This section examines $\mathscr{F}_{k,a}$ and its kernel $B_{k,a}(x,y)$ in the rank-one case.

Suppose $N = 1$, $a > 0$, $k \geq 0$, and $2k > 1 - a$. Then, by the explicit formula of the kernel $\Lambda_{k,a}$ (see Proposition 4.29), followed by the formula (5.9), we get

$$\begin{aligned} B_{k,a}(x,y) &= e^{i\frac{\pi}{2}(\frac{2k+a-1}{a})}\Lambda_{k,a}(x,y; i\frac{\pi}{2}) \\ &= \Gamma\Big(\frac{2k+a-1}{a}\Big)\Big(\widetilde{J}_{\frac{2k-1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big) + \frac{xy}{(ai)^{\frac{2}{a}}}\widetilde{J}_{\frac{2k+1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big)\Big), \end{aligned} \tag{5.18}$$

where $\widetilde{J}_\nu(w) = \widetilde{I}_\nu(-iw)$ is the normalized Bessel function given in (5.16); here the branch of $i^{\frac{2}{a}}$ is chosen so that $1^{\frac{2}{a}} = 1$. Thus, for $a > 0$, $k \in \mathbb{R}^+$ such that $2k > 1 - a$, and $f \in L^2(\mathbb{R}, |x|^{2k+a-2}dx)$, the integral transform $\mathscr{F}_{k,a}$ takes the form

$$\mathscr{F}_{k,a}f(y) = 2^{-1}a^{-(\frac{2k-1}{a})}\int_{\mathbb{R}} f(x)\Big(\widetilde{J}_{\frac{2k-1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big) + \frac{xy}{(ai)^{\frac{2}{a}}}\widetilde{J}_{\frac{2k+1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big)\Big)|x|^{2k+a-2}dx.$$

Remark 5.13. *If we set*

$$\begin{aligned} B^{\mathrm{even}}_{k,a}(x,y) &:= \frac{1}{2}[B_{k,a}(x,y) + B_{k,a}(x,-y)] \\ &= \Gamma\Big(\frac{2k+a-1}{a}\Big)\widetilde{J}_{\frac{2k-1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big). \end{aligned}$$

Then, the transform $\mathscr{F}_{k,a}(f)$ of an even function f on the real line specializes to a Hankel type transform on $\mathbb{R}_+$.

Let us find the formula (5.18) by an alternative approach. First, for general dimension $\mathbb{R}^N$, by composing (5.9), (4.52), and (5.15), we have

$$B_{k,a}(x,y) = \Big(\widetilde{V}_k\mathscr{I}\Big(\frac{2}{a}, \frac{2\langle k\rangle + N - 2}{2}; \frac{2(rs)^{\frac{a}{2}}}{ai}; \cdot\Big)\Big)(\langle\omega,\eta\rangle)$$

in the polar coordinates $x = r\omega$, $y = s\eta$. Furthermore, in the $N = 1$ case, a closed integral formula of the Dunkl intertwining operator V_k is known:

$$(V_kf)(x) = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}\int_{-1}^{1} f(tx)(1+t)(1-t^2)^{k-1}dt, \tag{5.19}$$

see [10, Theorem 5.1]. Hence, we might expect that the formula (5.18) for the kernel $B_{k,a}(x,y)$ could be recovered directly by using the integral formula (5.19) of V_k. In fact this is the case. To see this, we shall carry out a compution of the following integral:

$$B_{k,a}(x,y) = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}\int_{-1}^{1}\mathscr{I}\Big(\frac{2}{a},\frac{2k-1}{2},\frac{2(rs)^{\frac{a}{2}}}{ai};t\langle\omega,\eta\rangle\Big)(1+t)(1-t^2)^{k-1}dt \tag{5.20}$$

for $x=r\omega$, $y=s\eta$ ($r,s>0$, $\omega,\eta=\pm1$).

We notice that the summation (4.42) for $\mathscr{I}(b,\nu;w;t)$ is taken over $m=0$ and 1 if $N=1$. Hence we have

$$\mathscr{I}\Big(\frac{2}{a},\frac{2k-1}{2};\frac{2(rs)^{\frac{a}{2}}}{ai};t\langle\omega,\eta\rangle\Big) = \Gamma\Big(\frac{2k+a-1}{a}\Big)\Big(\widetilde{J}_{\frac{2k-1}{a}}\Big(\frac{2|xy|^{\frac{a}{2}}}{a}\Big)+\frac{(2k+1)txy}{(ai)^{\frac{2}{a}}}\widetilde{J}_{\frac{2k+1}{a}}\Big(\frac{2|xy|^{\frac{a}{2}}}{a}\Big)\Big).$$

On the other hand, by using the integral expression of the Beta function and the duplication formula (4.37) of the Gamma function, we have

$$\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}\int_{0}^{1}t^m(1+t)(1-t^2)^{k-1}dt = \begin{cases}1 & (m=0),\\ \dfrac{1}{2k+1} & (m=1).\end{cases}$$

Substituting these formulas into the right-hand side of (5.20) we have completed an alternative proof of (5.18).

5.5. **Master Formula and its applications.**

5.5.1. *Master Formula.*

We state the following two reproducing properties of the kernel $B_{k,a}$ of basic importance.

Theorem 5.14. (Master Formula) *Suppose $a>0$ and k is a non-negative multiplicity function satisfying* $2\langle k\rangle+N>\max(1,2-a)$.

1) *For $x,y\in\mathbb{R}^N$, we have*

$$\begin{aligned} &c_{k,a}\int_{\mathbb{R}^N}\exp\Big(\frac{i}{a}\|u\|^a\Big)B_{k,a}(x,u)B_{k,a}(u,y)\vartheta_{k,a}(u)du\\ &= e^{i\pi(\frac{2\langle k\rangle+N+a-2}{a})}\exp\Big(-\frac{i}{a}(\|x\|^a+\|y\|^a)\Big)B_{k,a}(x,y). \end{aligned}\tag{5.21}$$

2) *Let p be a homogeneous polynomial on $\mathbb{R}^N$ of degree m. Then we have*

$$\begin{aligned} &c_{k,a}\int_{\mathbb{R}^N}\exp\Big(-\frac{1}{a}\|u\|^a\Big)\Big(\exp\Big(-\frac{1}{2a}\|\cdot\|^{2-a}\Delta_k\Big)p\Big)(u)B_{k,a}(x,u)\vartheta_{k,a}(u)du\\ &= e^{-\frac{im\pi}{a}}\exp\Big(-\frac{1}{a}\|x\|^a\Big)\Big(\exp\Big(-\frac{1}{2a}\|\cdot\|^{2-a}\Delta_k\Big)p\Big)(x). \end{aligned}\tag{5.22}$$

Remark 5.15. *For $a=2$, the reproducing properties* (5.21) *and* (5.22) *were previously proved in Dunkl* [11, Theorem 3.2 and Proposition 2.1]. *In that case, theses properties played a crucial role in studying Dunkl analogues of Hermite polynomials (see* [51, Section 3]*), the properties of the heat kernel associated with the heat equation for the Dunkl operators (see* [51, Section 4]*), and in the construction of generalized Fock spaces (see* [4, Section 3]*).*

5.5.2. *Proof of Theorem 5.14.*
We begin with the proof of (5.21). From the semigroup law

$$\Omega_{k,a}(\gamma_{z_1})\Omega_{k,a}(\gamma_{z_2}) = \Omega_{k,a}(\gamma_{z_1+z_2}), \quad \text{for } \gamma_{z_1}, \gamma_{z_2} \in \widetilde{\Gamma(W)},$$

the integral representation of $\Omega_{k,a}(\gamma_z)$ (see Theorem 4.23) yields

$$c_{k,a} \int_{\mathbb{R}^N} \Lambda_{k,a}(x, u; i\frac{\pi}{4})\Lambda_{k,a}(u, y; i\frac{\pi}{4})\vartheta_{k,a}(u)du = \Lambda_{k,a}(x, y; i\frac{\pi}{2}). \tag{5.23}$$

We set

$$\mu := 2\langle k\rangle + N + a - 2.$$

In view of (4.49), a simple computation shows

$$\frac{h_{k,a}(r, s; \frac{\pi i}{4}; t)}{h_{k,a}(2^{\frac{1}{a}} r, s; \frac{\pi i}{2}; t)} = 2^{\frac{\mu}{2a}} \exp\Big(\frac{i}{a}(r^a + s^a)\Big).$$

Applying $\widetilde{V}_k$, and using (4.52), we get

$$\begin{aligned} \Lambda_{k,a}\Big(x, u; \frac{\pi i}{4}\Big) &= 2^{\frac{\mu}{2a}} \exp\Big(\frac{i}{a}(\|x\|^a + \|u\|^a)\Big)\Lambda_{k,a}\Big(2^{\frac{1}{a}} x, u; \frac{\pi i}{2}\Big) \\ &= (2e^{-\pi i})^{\frac{\mu}{2a}} \exp\Big(\frac{i}{a}(\|x\|^a + \|u\|^a)\Big)B_{k,a}(2^{\frac{1}{a}} x, u). \end{aligned} \tag{5.24}$$

In the second equality, we have used (5.9). By substituting (5.24) and (5.9) into (5.23), we get

$$c_{k,a} \int_{\mathbb{R}^N} \exp\Big(\frac{2i}{a}\|u\|^a\Big)B_{k,a}(2^{\frac{1}{a}} x, u)B_{k,a}(2^{\frac{1}{a}} u, y)\vartheta_{k,a}(u)du = 2^{-\frac{\mu}{a}} e^{i\pi\frac{\mu}{a}} \exp\Big(-\frac{i}{a}(\|x\|^a + \|y\|^a)\Big)B_{k,a}(x, y).$$

Since $B_{k,a}(2^{\frac{1}{a}} x, u) = B_{k,a}(x, 2^{\frac{1}{a}} u)$ (see Theorem 5.9 1)) and $\vartheta_{k,a}(u)du$ is homogeneous degree $N + 2\langle k\rangle + a - 2 = \mu$, the left-hand side equals

$$2^{-\frac{\mu}{a}} c_{k,a} \int_{\mathbb{R}^N} \exp\Big(\frac{i}{a}\|u\|^a\Big)B_{k,a}(x, u)B_{k,a}(u, y)\vartheta_{k,a}(u)du.$$

Hence, (5.21) is proved.

The remaining part of this subsection is devoted to the proof of the second statement of Theorem 5.14.

We recall from (3.3) and (3.6) that

$$\begin{aligned} \mathbb{E}^+_{k,a} &= \omega_{k,a}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{i}{a}\|x\|^a, \\ \mathbb{E}^-_{k,a} &= \omega_{k,a}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{i}{a}\|x\|^{2-a}\Delta_k \end{aligned}$$

are infinitesimal generators of the unitary representation $\Omega_{k,a}$ of $S\widetilde{L(2, \mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

We set

$$c_0 := \operatorname{Exp} i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \operatorname{Exp} \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.25}$$

and introduce the operator

$$\mathscr{B}_{k,a} := \exp(i\,\mathbb{E}^+_{k,a})\exp\Big(\frac{i}{2}\mathbb{E}^-_{k,a}\Big) = \exp\Big(-\frac{1}{a}\|x\|^a\Big)\exp\Big(-\frac{1}{2a}\|x\|^{2-a}\Delta_k\Big). \tag{5.26}$$

Then, the following identity in $\mathfrak{sl}_2$,

$$\operatorname{Ad}(c_0)\,\mathbf{h} = \begin{pmatrix}\frac{1}{2} & i\\ \frac{i}{2} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & i\\ \frac{i}{2} & 1\end{pmatrix}^{-1} = -i\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \mathbf{k}, \tag{5.27}$$

leads us to the identity of operators:

$$\mathscr{B}_{k,a}\circ\omega_{k,a}(\mathbf{h}) = \omega_{k,a}(\mathbf{k})\circ\mathscr{B}_{k,a}. \tag{5.28}$$

Since $\mathbb{H}_{k,a} = \omega_{k,a}(\mathbf{h})$ acts on homogeneous functions as scalar (see (3.3)), we know a priori that homogeneous functions applied by $\mathscr{B}_{k,a}$ are eigenfunctions of $\omega_{k,a}(\mathbf{k})$. Here is an explicit formula:

Proposition 5.16. *For $\ell, m \in \mathbb{N}$ and $p \in \mathscr{H}^m_k(\mathbb{R}^N)$,*

$$\mathscr{B}_{k,a}(p(x)\|x\|^{a\ell}) = \Big(-\frac{a}{2}\Big)^{\ell}\ell!\,\Phi^{(a)}_{\ell}(p,x).$$

Proof. We recall from Lemma 3.7 that the linear map

$$T_a : C^\infty(\mathbb{R}^N)\otimes C^\infty(\mathbb{R}_+)\to C^\infty(\mathbb{R}^N\setminus\{0\}),\quad (p,\psi)\mapsto p(x)\psi(\|x\|^a)$$

satisfies the following identity on $\mathscr{H}^m_k(\mathbb{R}^N)\otimes C^\infty(\mathbb{R}_+)$:

$$\exp\Big(\frac{i}{2}\mathbb{E}^-_{k,a}\Big)\circ T_a = T_a\circ\Big(\mathrm{id}\otimes\exp\Big(-\frac{a}{2}\Big(r\frac{d^2}{dr^2}+(\lambda_{k,a,m}+1)\frac{d}{dr}\Big)\Big)\Big). \tag{5.29}$$

Applying (5.29) to $p\otimes r^\ell$, and using Theorem 3.11, we get

$$\begin{aligned}\exp\Big(\frac{i}{2}\mathbb{E}^-_{k,a}\Big)\circ T_a(p\otimes r^\ell)(x) &= T_a\Big(p\otimes\Big(-\frac{a}{2}\Big)^{\ell}\ell!\,L^{(\lambda_{k,a,m})}_{\ell}\Big(\frac{2}{a}r\Big)\Big)(x)\\ &= \Big(-\frac{a}{2}\Big)^{\ell}\ell!\,p(x)L^{(\lambda_{k,a,m})}_{\ell}\Big(\frac{2}{a}\|x\|^a\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{B}_{k,a}(p(x)\|x\|^{a\ell}) &= \exp(i\,\mathbb{E}^+_{k,a})\exp\Big(\frac{i}{2}\mathbb{E}^-_{k,a}\Big)T_a(p\otimes r^\ell)\\ &= \Big(-\frac{a}{2}\Big)^{\ell}\ell!\,p(x)\exp\Big(-\frac{1}{a}\|x\|^a\Big)L^{(\lambda_{k,a,m})}_{\ell}\Big(\frac{2}{a}\|x\|^a\Big)\\ &= \Big(-\frac{a}{2}\Big)^{\ell}\ell!\,\Phi^{(a)}_{\ell}(p,x).\end{aligned}$$

Thus, Proposition 5.16 has been proved. ◻

Remark 5.17. *Let $p\in\mathscr{H}^m_k(\mathbb{R}^N)$. By* (3.3)*, $\omega_{k,a}(\mathbf{h})$ acts on $p(x)\|x\|^{a\ell}$ by the multiplication of the scalar $\lambda_{k,a,m}+1+2\ell$. Hence, $\omega_{k,a}(\mathbf{k})$ acts on $\Phi^{(a)}_{\ell}(p,x)$ as the same scalar $\lambda_{k,a,m}+1+2\ell$. This gives an alternative proof of the formula* (3.33 a) *in Theorem 3.19.*

We now introduce the vector space

$$\mathscr{P}_a(\mathbb{R}^N) := \mathbb{C}\text{-span}\{p(x)\|x\|^{a\ell} : p \in \mathscr{H}_k^m(\mathbb{R}^N) \text{ for some } m \in \mathbb{N}, \ell \in \mathbb{N}\}. \tag{5.30}$$

For $a = 2$, $\mathscr{P}_2(\mathbb{R}^N)$ coincides with the space $\mathscr{P}(\mathbb{R}^N)$ of polynomials on $\mathbb{R}^N$ owing to the following algebraic direct sum decomposition (see [4, Theorem 5.3]):

$$\mathscr{P}(\mathbb{R}^N) \simeq \bigoplus_{m=0}^{\infty} \bigoplus_{\ell=0}^{[\frac{m}{2}]} \|x\|^{2\ell} \mathscr{H}_k^{m-2\ell}(\mathbb{R}^N).$$

We introduce an endomorphism of $\mathscr{P}_a(\mathbb{R}^N)$, to be denoted by $(e^{-i\frac{\pi}{a}})^*$, as

$$\left(e^{-i\frac{\pi}{a}}\right)^*\left(p(x)\|x\|^{a\ell}\right) := e^{-i(\ell+\frac{m}{a})\pi} p(x)\|x\|^{a\ell}, \quad \text{for } p \in \mathscr{H}_k^m(\mathbb{R}^N) \text{ and } \ell \in \mathbb{N}. \tag{5.31}$$

Remark 5.18. *The notation $(e^{-i\frac{\pi}{a}})^*$ stands for the 'pull-back of functions' on the complex vector space $\mathbb{C}^N$ given by*

$$\left(e^{-i\frac{\pi}{a}}\right)^* f(z) = f\left(e^{-i\frac{\pi}{a}} z\right).$$

However, taking branches of multi-valued functions into account, we should note that $\left(e^{-i\frac{\pi}{a}}\right)^* \neq \mathrm{id}$ *for $a = \frac{1}{2}$.*

The next proposition is needed for later use.

Proposition 5.19. *For $a > 0$, the following diagram commutes*

$$\begin{array}{ccc}
\mathscr{P}_a(\mathbb{R}^N) & \xrightarrow{\mathscr{B}_{k,a}} & L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \\
{\scriptstyle (e^{-i\frac{\pi}{a}})^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{F}_{k,a}} \\
\mathscr{P}_a(\mathbb{R}^N) & \xrightarrow{\mathscr{B}_{k,a}} & L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)
\end{array}$$

Proof. The identity (5.28) in $\mathfrak{sl}_2$ lifts to the identity

$$\mathscr{B}_{k,a} \circ \Omega_{k,a}(\mathrm{Exp}\, t\,\mathbf{h}) = \Omega_{k,a}(\mathrm{Exp}\, t\,\mathbf{k}) \circ \mathscr{B}_{k,a},$$

and in particular

$$\mathscr{B}_{k,a} \circ \Omega_{k,a}(\mathrm{Exp}\, \frac{\pi}{2i}\mathbf{h}) = \Omega_{k,a}(\mathrm{Exp}\, \frac{\pi}{2i}\mathbf{k}) \circ \mathscr{B}_{k,a},$$

on $\mathscr{P}_a(\mathbb{R}^N)$ where the both-hand sides make sense. In terms of the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ (see (5.2)), we get

$$\mathscr{B}_{k,a} \circ \exp\Big(\frac{\pi i}{2}\frac{2\langle k\rangle + N + a - 2}{a}\Big)\,\Omega_{k,a}\Big(\mathrm{Exp}\, \frac{\pi}{2i}\mathbf{h}\Big) = \mathscr{F}_{k,a} \circ \mathscr{B}_{k,a}.$$

On the other hand, we recall from (3.6) that

$$\omega_{k,a}(\mathbf{h}) = \frac{2}{a}\sum_{j=1}^{N} x_j\partial_j + \frac{N + 2\langle k\rangle + a - 2}{a},$$

and therefore its lift to the group representation is given by

$$(\Omega_{k,a}(\mathrm{Exp}\, t\,\mathbf{h}) f)(x) = \exp\Big(\frac{N + 2\langle k\rangle + a - 2}{a} t\Big) f(e^{\frac{2t}{a}} x). \tag{5.32}$$

Substituting $t = \frac{\pi}{2i}$, we get $\mathscr{B}_{k,a} \circ \left(e^{-i\frac{\pi}{a}}\right)^* = \mathscr{F}_{k,a} \circ \mathscr{B}_{k,a}$. This completes the proof of Proposition 5.19 □

When $k \equiv 0$ and $a = 2$, $\mathscr{B}_{0,2}$ coincides with the inverse of the Segal–Bargmann transform restricted to $\mathscr{P}(\mathbb{R}^N) = \mathscr{P}_{0,2}(\mathbb{R}^N)$ (cf. [18, p. 40]). We may think of $\mathscr{B}_{k,a}$ as a (k, a)-generalized Segal–Bargmann transform. We are ready to prove the second statement of Theorem 5.14.

Proof of Theorem 5.14 2). In view of Proposition 5.19, we have $\mathscr{F}_{k,a} \circ \mathscr{B}_{k,a}(p) = \mathscr{B}_{k,a} \circ (e^{-i\frac{\pi}{a}})^*(p)$. Since $(e^{-i\frac{\pi}{a}})^* p(x) = e^{-\frac{im\pi}{a}} p(x)$ for a homogeneous polynomial of degree m, we get

$$\mathscr{F}_{k,a} \circ \mathscr{B}_{k,a}(p) = e^{-\frac{im\pi}{a}} \mathscr{B}_{k,a}(p).$$

Hence, the reproducing property (5.22) is proved. □

5.5.3. *Application of Master Formula.*
As an immediate consequence of Master Formula (see Theorem 5.14), we have:

Corollary 5.20. (Hecke type identity) *If in addition to the assumption in Theorem 5.14 2), the polynomial p is k-harmonic of degree m, then* (5.22) *reads*

$$\mathscr{F}_{k,a}(e^{-\frac{\|\cdot\|^a}{a}} p)(\xi) = e^{-i\frac{\pi}{a}m} e^{-\frac{1}{a}\|\xi\|^a} p(\xi). \tag{5.33}$$

Corollary 5.20 may be regarded as a Hecke type identity for the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$. An alternative way to prove this identity would be to substitute 0 for ℓ in (5.3).

The identity (5.33) is a particular case of Theorem 5.21 below. For this, we will denote by $\boldsymbol{H}_{a,\nu}$ the classical Hankel transform of one variable defined by

$$\boldsymbol{H}_{a,\nu}(\psi)(s) := \int_0^\infty \psi(r) \widetilde{J}_\nu\Big(\frac{2}{a}(rs)^{\frac{a}{2}}\Big) r^{a(\nu+1)-1} dr, \tag{5.34}$$

for a function ψ defined on $\mathbb{R}_+$. Here, $\widetilde{J}_\nu$ is the normalized Bessel function $\widetilde{J}_\nu(w) = (\frac{w}{2})^{-\nu} J_\nu(w)$ (see (4.9)). Then the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ satisfies the following identity:

Theorem 5.21. (Bochner type identity) *If $f \in (L^1 \cap L^2)(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is of the form $f(x) = p(x)\psi(\|x\|)$ for some $p \in \mathscr{H}_k^m(\mathbb{R}^N)$ and a one-variable function ψ on $\mathbb{R}_+$, then*

$$\mathscr{F}_{k,a}(f)(\xi) = a^{-(\frac{2m+2\langle k\rangle+N-2}{a})} e^{-i\frac{\pi}{a}m} p(\xi)\, \boldsymbol{H}_{a,\frac{2m+2\langle k\rangle+N-2}{a}}(\psi)(\|\xi\|).$$

In particular, if f is radial, then $\mathscr{F}_{k,a}(f)$ is also radial.

Remark 5.22. *The original Bochner identity for the Euclidean Fourier transform corresponds to the case $a = 2$ and $k \equiv 0$. For $a = 2$ and $k > 0$, Theorem 5.21 corresponds to the Bochner identity for the Dunkl transform which was proved in* [2]. *For $a = 1$ and $k \equiv 0$ it is the Bochner identity for the Hankel-type transform on $\mathbb{R}^N$ (see* [36]*).*

Proof of Theorem 5.21. It follows from (4.20) that

$$\begin{aligned}\Lambda_{k,a}^{(m)}\Big(r, s; \frac{\pi i}{2}\Big) &= \exp\Big(-\frac{\pi i}{2}(\lambda_{k,a,m} + 1)\Big)(rs)^{-\langle k\rangle - \frac{N}{2}+1} J_{\lambda_{k,a,m}}\Big(\frac{2}{a}(rs)^{\frac{a}{2}}\Big) \\ &= a^{-\lambda_{k,a,m}}(rs)^m \exp\Big(-\frac{\pi i}{2}(\lambda_{k,a,m} + 1)\Big)\widetilde{J}_{\lambda_{k,a,m}}\Big(\frac{2}{a}(rs)^{\frac{a}{2}}\Big).\end{aligned} \tag{5.35}$$

We set $\psi_m(r) := r^m\psi(r)$. Since p is homogeneous of degree m, we have

$$p\left(\frac{x}{\|x\|}\right)\psi_m(\|x\|) = p(x)\psi(\|x\|).$$

From the definition of the unitary operator $\Omega^{(m)}_{k,a}(\gamma_z)$ (see (4.3)), we get

$$\begin{aligned}\Omega_{k,a}(\gamma_z)f(x) &= p(x)\|x\|^{-m}\Omega^{(m)}_{k,a}(\gamma_z)\psi_m(\|x\|)\\ &= p(x)\|x\|^{-m}\int_0^\infty \Lambda^{(m)}_{k,a}(\|x\|, s; z)\psi_m(s)s^{2\langle k\rangle+N+a-3}ds.\end{aligned}$$

Substituting (5.35) into the above formula with $z = \frac{\pi i}{2}$, we get

$$\begin{aligned}\Omega_{k,a}(\gamma_{\frac{\pi i}{2}})f(x) &= a^{-\lambda_{k,a,m}}\exp\Big(-\frac{\pi i}{2}(\lambda_{k,a,m}+1)\Big)p(x)\int_0^\infty \widetilde{J}_{\lambda_{k,a,m}}\left(\frac{2}{a}(\|x\|s)^{\frac{a}{2}}\right)\psi(s)s^{2m+2\langle k\rangle+N+a-3}ds\\ &= a^{-\lambda_{k,a,m}}\exp\Big(-\frac{\pi i}{2}(\lambda_{k,a,m}+1)\Big)p(x)\boldsymbol{H}_{a,\lambda_{k,a,m}}(\psi)(\|x\|).\end{aligned}$$

Now, Theorem 5.21 follows from (5.2). ◻

5.6. **DAHA and SL_2-action.**

In this subsection we discuss some link between the representation $\Omega_{k,a}$ of $\widetilde{SL(2,\mathbb{R})}$ in the $a = 2$ case and the (degenerate) rational DAHA (double affine Hecke algebra). To be more precise, we shall see that our representation $\Omega_{k,2}$ of $\widetilde{SL(2,\mathbb{R})}$ induces the representation of $SL(2,\mathbb{C})$ on the algebra generated by Dunkl's operators, multiplication operators, and the Coxeter group (see (5.42) below). This induced action on the operators coincides essentially with a special case of the $SL(2,\mathbb{Z})$-action discovered by Cherednik [5] and that of the $SL(2,\mathbb{C})$-action by Etingof and Ginzburg [17]. Note that our approach (but not the result) is new in that we use our action on functions to derive the action on the operators in the Hecke algebra. The authors are grateful to E. Opdam for bringing their attention to this link.

We begin with an observation that if Ω is a representation of a group G on a vector space W then we can define an automorphism of the associative algebra $\operatorname{End}(W)$ by

$$A \mapsto \Omega(g)A\Omega(g)^{-1}, \quad g \in G. \tag{5.36}$$

We shall consider this induced action for $G = \widetilde{SL(2,\mathbb{R})}$, $\Omega = \Omega_{k,2}$ (see Theorem 3.30), $W =$ the vector space consisting of appropriate functions on $\mathbb{R}^N$.

Remark 5.23. *We do not specify the class of functions here. Instead, we shall use the formula* (5.36) *to define algebraically the G-action on a certain subspace of* $\operatorname{End}(W)$. *The point here is that the G-action on such a subspace will be well-defined even when the group G may not preserve W.*

We begin with a basic fact on Dunkl operators on $\mathbb{R}^N$. For $\xi \in \mathbb{R}^N$, we define the multiplication operator M_ξ by

$$M_\xi f(x) := \langle \xi, x\rangle f(x).$$

Choose an orthonormal basis $\xi_1, \dots, \xi_N$ in $\mathbb{R}^N$. As in Section 2.2, we will use the abbreviation $T_i(k)$ for Dunkl operators $T_{\xi_i}(k)$, and M_j for M_{ξ_j}. Then we have the following commutation relations:

$$[T_i(k), M_j] = \delta_{ij} + 2 \sum_{\alpha \in \mathscr{R}^+} k_\alpha \frac{\langle \alpha, \xi_i \rangle \langle \alpha, \xi_j \rangle}{\|\alpha\|^2} r_\alpha, \quad \text{for any } 1 \le i, j \le N. \tag{5.37}$$

Since the formula (5.37) is symmetric with respect to i and j, we have:

$$[T_i(k), M_j] = [T_j(k), M_i] \quad \text{for any } 1 \le i, j \le N. \tag{5.38}$$

Furthermore, we have the following formulas:

Lemma 5.24. *Let $\xi \in \mathbb{R}^N$ and $s \in \mathbb{C}$.*

1) $[\Delta_k, M_\xi] = 2T_\xi(k)$.
2) $e^{s\Delta_k} M_\xi e^{-s\Delta_k} = M_\xi + 2sT_\xi(k)$.

The first statement is due to Dunkl [9, Proposition 2.2], *but we give its proof below for the reader's convenience.*

Proof. 1) It is sufficient to prove the formula for $\xi = \xi_j$ ($j = 1, \ldots, N$).

By using (5.37), we have

$$\begin{aligned}
[T_i^2(k), M_{\xi_j}] &= T_i(k)[T_i(k), M_j] + [T_i(k), M_j]T_i(k) \\
&= 2\delta_{ij} T_i(k) + 2 \sum_{\alpha \in \mathscr{R}^+} k_\alpha \frac{\langle \alpha, \xi_i \rangle \langle \alpha, \xi_j \rangle}{\|\alpha\|^2} (T_i(k) r_\alpha + r_\alpha T_i(k)).
\end{aligned}$$

Summing them up over i, and using the following relations:

$$\begin{aligned}
&\sum_{i=1}^N \langle \alpha, \xi_i \rangle T_i(k) = T_\alpha(k), \\
&T_\alpha(k) r_\alpha + r_\alpha T_\alpha(k) = 0, \quad \text{(see (D1) in Section 2.1)},
\end{aligned}$$

we get

$$[\Delta_k, M_{\xi_j}] = 2T_j(k).$$

2) The second statement is straightforward from the first statement. ◻

Let us consider the induced action of G on $\mathrm{End}(W)$ (see Remark 5.23).

Proposition 5.25. *We fix a non-zero $\xi \in \mathbb{R}^N$.*

1) *The induced action of* $\Omega_{k,2}$ *by* (5.36) *preserves the two dimensional subspace*

$$\mathbb{C}^2_\xi := \mathbb{C}M_\xi + \mathbb{C}T_\xi(k).$$

2) *The resulting representation of $\widetilde{SL(2, \mathbb{R})}$ on $\mathbb{C}^2_\xi$ descends to $SL(2, \mathbb{R})$, and extends holomorphically to $SL(2, \mathbb{C})$.*
3) *Via the basis $\{M_\xi, T_\xi(k)\}$, the representation of $SL(2, \mathbb{C})$ on $\mathbb{C}^2_\xi$ is given by*

$$\varphi : SL(2, \mathbb{C}) \to GL_{\mathbb{C}}(\mathbb{C}^2_\xi), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -iB \\ iC & D \end{pmatrix}. \tag{5.39}$$

Proof. 1) Since $\widetilde{SL(2,\mathbb{R})}$ is generated by $\operatorname{Exp}(t\mathbf{e}^+)$ and $\operatorname{Exp}(t\mathbf{e}^-)$ ($t \in \mathbb{R}$), it is sufficient to prove that the subspace $\mathbb{C}M_\xi + \mathbb{C}T_\xi(k)$ is stable by the induced action of these generators. In light of the formula (3.3), we have

$$\Omega_{k,2}(\operatorname{Exp} t\mathbf{e}^+) = \operatorname{Exp}(t\mathbb{E}^+_{k,2}) = \exp\left(\frac{it}{2}\|x\|^2\right).$$

Obviously, this action commutes with the multiplication operator M_ξ. On the other hand, applying (2.21) with $a = 2$ and $\lambda = -\frac{it}{2}$, we get

$$\exp\left(\frac{it}{2}\|x\|^2\right) \circ T_\xi(k) \circ \exp\left(-\frac{it}{2}\|x\|^2\right) = T_\xi(k) - itM_\xi.$$

Hence, $\operatorname{Exp}(t\mathbb{E}^+_{k,2})$ preserves the two-dimensional subspace $\mathbb{C}M_\xi + \mathbb{C}T_\xi(k)$, and its action is given by the following matrix form

$$\operatorname{Exp}(t\mathbf{e}^+) \mapsto \begin{pmatrix} 1 & -it \\ 0 & 1 \end{pmatrix} \tag{5.40}$$

with respect to the basis $\{M_\xi, T_\xi(k)\}$.

Next, we consider the action

$$\Omega_{k,2}(\operatorname{Exp}(t\mathbf{e}^-)) = \operatorname{Exp}(t\mathbb{E}^-_{k,2}) = \exp\left(\frac{it}{2}\Delta_k\right).$$

Obviously, it commutes with Dunkl's operator $T_\xi(k)$. On the other hand, applying Lemma 5.24 2) with $s = \frac{it}{2}$, we get

$$\exp\left(\frac{it}{2}\Delta_k\right) \circ M_\xi \circ \exp\left(-\frac{it}{2}\Delta_k\right) = M_\xi + itT_\xi(k).$$

Hence, $\operatorname{Exp}(t\mathbb{E}^-_{k,2})$ also preserves the subspace $\mathbb{C}M_\xi + \mathbb{C}T_\xi(k)$, and its action is given as

$$\operatorname{Exp}(t\mathbf{e}^-) \mapsto \begin{pmatrix} 1 & 0 \\ it & 1 \end{pmatrix}. \tag{5.41}$$

Thus, we have proved the first statement.

2) The center of $\widetilde{SL(2,\mathbb{R})}$ consists of the elements $\operatorname{Exp}(in\pi\mathbf{k})$ ($n \in \mathbb{Z}$) (see (3.38)). Let us compute the action of $\operatorname{Exp}(t\mathbf{k})$ on $\mathbb{C}^2_\xi = \mathbb{C}M_\xi + \mathbb{C}T_\xi(k)$. For this, we recall from (5.27) that

$$\operatorname{Exp}(t\mathbf{k}) = c_0 \operatorname{Exp}(t\mathbf{h})c_0^{-1} \quad \text{for } t \in \mathbb{C}.$$

In view of the formulas (5.40) and (5.41), the element $c_0 = \operatorname{Exp} i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \operatorname{Exp} \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (see (5.25)) acts on $\mathbb{C}^2_\xi$ as

$$c_0 \mapsto \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{i}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{i}{2} & 1 \end{pmatrix}.$$

It follows readily from the formula

$$(\Omega_{k,2}(\operatorname{Exp}(t\mathbf{h}))f)(x) = \exp\left(\frac{N + 2\langle k\rangle}{2}t\right) f(e^t x)$$

(see (5.32)) that the action on $\mathrm{Exp}(t\mathbf{h})$ on $\mathbb{C}^2_\xi$ is given by

$$\mathrm{Exp}(t\mathbf{h}) \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Therefore, $\mathrm{Exp}(t\mathbf{k})$ acts on $\mathbb{C}^2_\xi$ by the formula:

$$\mathrm{Exp}(t\mathbf{k}) \mapsto \begin{pmatrix} \frac{1}{2} & i \\ \frac{i}{2} & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & i \\ \frac{i}{2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cosh(t) & -i\sinh(t) \\ i\sinh(t) & \cosh(t) \end{pmatrix}.$$

In particular, if $t = in\pi$, then $\mathrm{Exp}(in\pi\mathbf{k})$ acts as $(-1)^n$ id on $\mathbb{C}^2_\xi$. Thus, the action of $S\widetilde{L(2,\mathbb{R})}$ descends to $S\widetilde{L(2,\mathbb{R})}/2\mathbb{Z} \simeq SL(2,\mathbb{R})$. Then, clearly, this two-dimensional representation extends holomorphically to $SL(2,\mathbb{C})$. Hence, the second statement is proved.

3) Since a representation of $SL(2,\mathbb{C})$ is uniquely determined by the generators $\mathrm{Exp}(t\mathbf{e}^+)$ and $\mathrm{Exp}(t\mathbf{e}^-)$ ($t \in \mathbb{C}$), the third statement follows from (5.40) and (5.41). □

Let $\mathbb{H}$ be the algebra generated by

$$\{M_\xi, T_\xi : \xi \in \mathbb{R}^N\} \cup \mathfrak{C}, \tag{5.42}$$

where $\mathfrak{C}$ is the Coxeter group. Its defining relations are given by the commutativity of the Dunkl operators $T_\xi(k)$ (see (D2) in Section 2), the commutativity of the multiplication operators M_ξ, the commutation relations (5.37), and the following $\mathfrak{C}$-equivariance:

$$h \circ T_\xi(k) \circ h^{-1} = T_{h\xi}(k), \quad h \circ M_\xi \circ h^{-1} = M_{h\xi}, \quad \text{for any } h \in \mathfrak{C},\ \xi \in \mathbb{R}^N,$$

see [6, 17].

We recall from Proposition 5.25 3) that the matrix representation of the $SL(2,\mathbb{C})$-action on $\mathbb{C}^2_\xi = \mathbb{C}M_\xi + \mathbb{C}T_\xi(k)$ does not depend on $\xi \in \mathbb{R}^N \setminus \{0\}$. Then, a simple computation relied on (5.38) yields

$$[g \cdot T_i(k), g \cdot T_j(k)] = 0 = g \cdot [T_i(k), T_j(k)],$$
$$[g \cdot M_i, g \cdot M_j] = 0 = g \cdot [M_i, M_j],$$

for any $1 \le i, j \le N$ and for any $g \in SL(2,\mathbb{C})$. Likewise, we get from (5.37)

$$[g \cdot T_i(k), g \cdot M_j] = [T_i(k), M_j].$$

Furthermore, the representation $\Omega_{k,a}$ of $S\widetilde{L(2,\mathbb{R})}$ commutes with the action of the Coxeter group $\mathfrak{C}$. Therefore, the action of $SL(2,\mathbb{C})$ on $\mathbb{C}^2_\xi$ ($\xi \in \mathbb{R}^N \setminus \{0\}$) and the trivial action on the Coxeter group $\mathfrak{C}$ extends to an automorphism of $\mathbb{H}$ because all the defining relations of $\mathbb{H}$ are preserved by $SL(2,\mathbb{C})$.

Hence, we have proved:

Theorem 5.26. *The representation $\Omega_{k,2}$ of $S\widetilde{L(2,\mathbb{R})}$ induces the above action of $SL(2,\mathbb{C})$ on the algebra $\mathbb{H}$ as automorphisms.*

Remark 5.27. *The $SL(2,\mathbb{C})$-action on the algebra $\mathbb{H}$ is essentially the same with the one given in* [17, Corollary 5.3].

Remark 5.28. *As we saw in the proof of Proposition 5.25, the center* $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *of* $SL(2,\mathbb{C})$ *acts on* $\mathbb{C}^2_\xi$ *as* $-\mathrm{id}$. *Therefore,* $PSL(2,\mathbb{C})$ *acts on* $I\!H$ *as projective automorphisms.*

In order to compare the $SL(2,\mathbb{Z})$-action on $I\!H$ defined by Cherednik [5] we consider the following automorphism of $SL(2,\mathbb{C})$:

$$\iota : SL(2,\mathbb{C}) \to SL(2,\mathbb{C}), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -iB \\ iC & D \end{pmatrix}, \tag{5.43}$$

and twist the $SL(2,\mathbb{C})$-action on $I\!H$ (see Theorem 5.26) by ι. This means that the new action takes the form

$$\varphi \circ \iota : SL(2,\mathbb{C}) \to GL_{\mathbb{C}}(\mathbb{C}^2_\xi), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \tag{5.44}$$

on the generators $\mathbb{C}^2_\xi = \mathbb{C}M_\xi + \mathbb{C}T_\xi(k)$ (see (5.39)).

We write τ_1 and τ_2 for the automorphisms of $I\!H$ corresponding to the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $SL(2,\mathbb{Z})$. Then, by (5.44), τ_1 and τ_2 are given by

$$\tau_1 : M_\xi \mapsto M_\xi, \quad T_\xi \mapsto T_\xi - M_\xi, \quad h \mapsto h,$$
$$\tau_2 : T_\xi \mapsto T_\xi, \quad M_\xi \mapsto M_\xi - T_\xi, \quad h \mapsto h,$$

which coincide with the one given in [5].

Of particular importance in Cherednik [5] is the automorphism

$$\sigma := \tau_1 \tau_2^{-1} \tau_1 = \tau_2^{-1} \tau_1 \tau_2^{-1},$$

which corresponds to the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The automorphism σ is characterized by

$$\sigma(T_\xi) = -M_\xi, \quad \sigma(M_\xi) = T_\xi \quad \text{and} \quad \sigma(h) = h \quad \text{for all } h \in \mathfrak{C}.$$

From our view point, these automorphisms on $I\!H$ can be obtained as the conjugations of the action on the function space (see (5.36)). In view of the formulas (see (5.43)):

$$\iota \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix},$$
$$\iota \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix},$$
$$\iota \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \operatorname{Exp}\left(\frac{\pi i}{2}\mathbf{h}\right)\gamma_{\frac{\pi i}{2}},$$

we may interpret that τ_1, τ_2, and σ are given by the conjugations of

$$\Omega_{k,2}(\operatorname{Exp}(-i\mathbf{e}^+)) = \exp\Bigl(\frac{1}{2}\|x\|^2\Bigr),$$
$$\Omega_{k,2}(\operatorname{Exp}(i\mathbf{e}^-)) = \exp\Bigl(-\frac{1}{2}\Delta_k\Bigr),$$

$$\mathscr{F}^{re}_{k,2} := e^{\frac{1}{2}\|x\|^2} \circ \exp\Big(\frac{1}{2}\Delta_k\Big) \circ e^{\frac{1}{2}\|x\|^2} = \exp\Big(\frac{1}{2}\Delta_k\Big) \circ e^{\frac{1}{2}\|x\|^2} \circ \exp\Big(\frac{1}{2}\Delta_k\Big) \tag{5.45}$$
$$= \Omega_{k,2}\Big(\operatorname{Exp}\frac{\pi i}{2}\mathbf{h}\Big)\Omega_{k,2}(\gamma_{\frac{\pi i}{2}}),$$

respectively. Recalling the formulas:

$$(\Omega_{k,2}\Big(\operatorname{Exp}\frac{\pi i}{2}\mathbf{h}\Big)f)(x) = \exp\left(\frac{i\pi(N+2\langle k\rangle)}{4}\right)f(ix),$$
$$(\Omega_{k,2}(\gamma_{\frac{\pi i}{2}})f)(x) = \exp\left(-\frac{i\pi(N+2\langle k\rangle)}{4}\right)(\mathscr{F}_{k,2}f)(x),$$

we have

$$\mathscr{F}^{re}_{k,2}f(x) = \mathscr{F}_{k,2}f(ix).$$

Hence, σ may be interpreted as an algebraic version of the Dunkl transform. We notice that the formula (5.45) fits well into Master Formula (5.22) for $a = 2$, which we may rewrite as

$$c_{k,2}\int_{\mathbb{R}^N} e^{-\frac{1}{2}\|u\|^2}\big(\exp\Big(-\frac{1}{2}\Delta_k\Big)p)(u)B_{k,2}(ix,u)\prod_{\alpha\in\mathscr{R}}|\langle\alpha,u\rangle|^{k_\alpha}du = e^{\frac{1}{2}\|x\|^2}p(x).$$

5.7. The uncertainty inequality for the transform $\mathscr{F}_{k,a}$.

The Heisenberg uncertainty principle may be formulated by means of the so-called Heisenberg inequality for the Euclidean Fourier transform on $\mathbb{R}$. Loosely, the more a function is concentrated, the more its Fourier transform is spread. We refer the reader to an excellent survey [19] for various mathematical aspects of the Heisenberg uncertainty principle. In this section we extend the Heisenberg inequality to the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ on $\mathbb{R}^N$.

Let $\|\cdot\|_k$ be the L^2-norm with respect to the measure $\vartheta_{k,a}(x)dx$ on $\mathbb{R}^N$ (see (1.2)). The goal of this subsection is to prove the following multiplicative inequality:

Theorem 5.29. (Heisenberg type inequality) *For all $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ satisfies*

$$\Big\|\|\cdot\|^{\frac{a}{2}}f\Big\|_k\,\Big\|\|\cdot\|^{\frac{a}{2}}\mathscr{F}_{k,a}(f)\Big\|_k \geq \Big(\frac{2\langle k\rangle+N+a-2}{2}\Big)\|f\|_k^2. \tag{5.46}$$

The equality holds if and only if the function f is of the form $f(x) = \lambda\exp(-c\|x\|^a)$ for some $\lambda\in\mathbb{C}$ and $c\in\mathbb{R}_+$.

Remark 5.30. *The inequality* (5.46) *for $k \equiv 0$ and $a = 2$ is the original Heisenberg inequality for the Euclidean Fourier transform. The inequality for $k > 0$ and $a = 2$ is the Heisenberg type inequality for the Dunkl transform, which was proved first by Rösler* [50] *and then by Shimeno* [53]*. In physics terms we can think of the function $f(x) = \lambda\exp(-c\|x\|^a)$ where the equality holds in the above theorem as a ground state; indeed when $a = c = 1$, $N = 3$, and $k \equiv 0$, it is exactly the wave function for the Hydrogen atom with the lowest energy.*

In order to prove Theorem 5.29 we begin with the following additive inequality:

Lemma 5.31. (1) *For all $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$*

$$\Big\|\|\cdot\|^{\frac{a}{2}}f\Big\|_k^2 + \Big\|\|\cdot\|^{\frac{a}{2}}\mathscr{F}_{k,a}(f)\Big\|_k^2 \geq (2\langle k\rangle+N+a-2)\|f\|_k^2. \tag{5.47}$$

(2) *The equality holds in* (5.47) *if and only if* $f(x)$ *is a scalar multiple of* $\exp(-\frac{1}{a}\|x\|^a)$.

Proof. By Theorem 5.6(3) and Theorem 5.1(1), we get

$$\begin{aligned}\left\|\|\cdot\|^{\frac{a}{2}}\mathscr{F}_{k,a}f\right\|_k^2 &= \langle\!\langle \|x\|^a\mathscr{F}_{k,a}f, \mathscr{F}_{k,a}f\rangle\!\rangle_k \\ &= -\langle\!\langle \mathscr{F}_{k,a}(\|x\|^{2-a}\Delta_k f), \mathscr{F}_{k,a}f\rangle\!\rangle_k \\ &= -\langle\!\langle \|x\|^{2-a}\Delta_k f, f\rangle\!\rangle_k.\end{aligned}$$

Hence, the left-hand side of (5.47) equals

$$\langle\!\langle (\|x\|^a - \|x\|^{2-a}\Delta_k)f, f\rangle\!\rangle_k = \langle\!\langle -\Delta_{k,a}f, f\rangle\!\rangle_k. \tag{5.48}$$

It then follows from Corollary 3.22 that the self-adjoint operator $-\Delta_{k,a}$ has only discrete spectra, of which the minimum is $2\langle k\rangle + N - 2 + a$. Therefore, we have proved

$$(5.48) \geq (2\langle k\rangle + N - 2 + a)\|f\|_k^2.$$

Thus, the inequality (5.47) has been proved. Further, the equality holds if and only if f is an eigenfunction of $-\Delta_{k,a}$ corresponding to the minimum eigenvalue $2\langle k\rangle + N - 2 + a$, namely, f is a scalar multiple of $\exp(-\frac{1}{a}\|x\|^a)$ (i.e. by putting $\ell = m = 0$ in the formula (3.28) of $\Phi_\ell^{(a)}(p, x)$). Hence, Lemma 5.31 has been proved. ◻

Proof of Theorem 5.29. Now, for $c > 0$, we set $f_c(x) := f(cx)$. Using the fact that the density $\vartheta_{k,a}$ is homogeneous of degree $2\langle k\rangle + a - 2$, we get

$$\left\|\|\cdot\|^{\frac{a}{2}}f_c\right\|_k^2 = c^{-2\langle k\rangle - N - 2a + 2}\left\|\|\cdot\|^{\frac{a}{2}}f\right\|_k^2,$$

and

$$\|f_c\|_k^2 = c^{-2\langle k\rangle - N - a + 2}\|f\|_k^2.$$

Furthermore, we lift the formula in Theorem 5.9(1) to the formula

$$(\mathscr{F}_{k,a}f_c)(x) = c^{-(N+2\langle k\rangle + a - 2)}(\mathscr{F}_{k,a}f)\Big(\frac{x}{c}\Big),$$

from which we get

$$\left\|\|\cdot\|^{\frac{a}{2}}\mathscr{F}_{k,a}(f_c)\right\|_k^2 = c^{-2\langle k\rangle - N + 2}\left\|\|\cdot\|^{\frac{a}{2}}\mathscr{F}_{k,a}(f)\right\|_k^2.$$

Thus, if we substitute f_c for f in Lemma 5.31, we obtain

$$c^{-a}\left\|\|\cdot\|^{\frac{a}{2}}f\right\|_k^2 + c^a\left\|\|\cdot\|^{\frac{a}{2}}\mathscr{F}_{k,a}(f)\right\|_k^2 \geq (2\langle k\rangle + N + a - 2)\|f\|_k^2.$$

Obviously the minimum value of the left-hand side (as a function of $c \in \mathbb{R}_+$) is

$$2\left\|\|\cdot\|^{\frac{a}{2}}f\right\|_k \left\|\|\cdot\|^{\frac{a}{2}}\mathscr{F}_{k,a}(f)\right\|_k.$$

Hence, Theorem 5.29 has been proved. ◻

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